BIRATIONAL GEOMETRY OF GENUS ONE FIBRATIONS AND STABILITY OF PENCILS OF PLANE CURVES

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To my sister Paula

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ABSTRACT

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In the first part of this thesis we give a complete classification of relative log canonical models for genus one fibrations in dimensions two and three. More concretely, we generalize the work in [2] by considering both (i) the case where it is not assumed the existence of a section, but of a multisection instead; and (ii) the case of threefolds in one dimension higher. In the second part, we investigate the stability of pencils of plane curves in the sense of geometric invariant theory. One of our main results relates the stability of a pencil of plane curves $\mathcal{P}$ to the log canonical threshold of pairs $(\mathbb{P}^2, \mathcal{C}_d)$, where $\mathcal{C}_d$ is a curve in $\mathcal{P}$, thus extending an idea of Hacking [23] and Kim-Lee [27]. Part of our approach consists in observing that we can sometimes determine whether a pencil $\mathcal{P}$ is (semi)stable or not by looking at the stability of the curves lying on it. As a beautiful application, we completely describe the stability of Halphen pencils of index two – classical geometric objects first introduced by Halphen in 1882 [24]. Inspired by the work of Miranda in [40], we provide explicit stability criteria in terms of the geometry of their associated rational elliptic surfaces.

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Preface

The present thesis is divided into two parts, the unifying theme being the geometry of genus one fibrations (Definition I.2.0.1). We work over $\mathbb{C}$ throughout, and each part is intended to be self-contained. In particular, each part has its own introduction.

In Part I we give a complete classification of relative log canonical models (Definition I.3.1.1) for genus one fibrations in dimensions two and three, which is the first step in constructing their moduli spaces via the Minimal Model Program (MMP) as proposed by Kollár-Shepherd-Barron [37] (KSB compactification).

In Part II we investigate the stability of pencils of plane curves in the sense of geometric invariant theory (GIT). In particular, we give a complete description of the stability of Halphen pencils of index two (Definition II.3.1.4). Inspired by [40], we provide explicit stability criteria in terms of the geometry of their associated rational elliptic surfaces (RES). This part consists of the content of three papers [54–56].

The results obtained in this thesis naturally lead us to the question of how to relate the KSB and GIT moduli spaces for RES of index two. We plan to address this question in a future project.

Part I

Birational geometry of genus one fibrations

Chapter I.1

Introduction

One of the main questions in Birational Geometry consists in describing convenient birational models for algebraic varieties. By Chow's lemma, every algebraic variety is birational to some projective variety, and in fact (over a field of characteristic zero) Hironaka's theorem implies every algebraic variety is birational to a smooth projective variety. Therefore, it suffices to consider birational models for smooth projective varieties. In dimension one, each birational class contains a unique (up to isomorphism) smooth projective curve. In other words, if two smooth projective curves are birational, then they are isomorphic. In higher dimensions, however, this fails and leads to the notion of minimal models: is there a unique simplest algebraic variety in each birational class?

A similar but more general problem is to consider pairs (X, Δ), where X is a normal algebraic variety and $\Delta \subset X$ is a natural choice of divisor with only mild singularities. One can also consider a relative version of the latter problem, where a projective morphism $f : X \to S$ is part of the input data. This leads to the notion of relative log canonical (lc) models – the type of singularities in Δ that we allow are called log canonical singularities (Definition I.3.0.3).

In the present thesis we are interested in describing and classifying relative lc models for pairs (X, Δ), where X admits a genus one fibration $f : X \to S$ (Definition I.2.0.1) and the boundary divisor Δ is supported in a section or multisection for the fibration plus some weighted fiber(s). Note that these objects, hence Δ, are intrinsic to the genus one fibration.

We follow the ideas first introduced in [2] and [3] where the authors considered elliptically fibered surfaces with a section. We generalize their results by considering both the case where it is not assumed the existence of a section, but of a multisection instead, and the case of threefolds in one dimension higher. Our results build on Kodaira's classification of singular fibers (Table I.2.1), on Miranda's construction of smooth models for elliptic threefolds [41] and on a relative version of the abundance conjecture in dimensions two and three [14, 48].

One of our goals is to understand how these relative log canonical models vary with the choice of the weight. The results we obtain generalize the results in [2] and illustrate the fact that such models depend not only on the type of the marked fiber, but also on the geometry of the intersection between the section/multisection and the marked fiber. Another interesting feature is the appearance of an important birational invariant called log canonical threshold (Definition I.3.0.4).

Our exposition is organized as follows: Chapter I.2 describes the background material needed on genus one fibrations. In Chapter I.3 we recall the basic notions concerning log canonical pairs and we present a general overview of the log Minimal Model Program (log MMP) in the relative setting. Next, in Chapter I.4 we run the relative log minimal models program for genus one fibrations in dimension two. In Chapter I.5 we give a classification (Theorem I.5.0.1) of relative log canonical models of elliptic surface pairs $(f : X \to C, a_M M + aF)$ of index $d_X = 2$, where M is a multisection of degree equals d_X and $a_M = 1/d_X = 1/2$. Finally, in Chapter I.6 we provide a classification (Theorem I.6.0.15) of relative lc models of certain elliptic threefold pairs $(f : X \to S, S + aF_1 + bF_2)$, where X is a smooth model as constructed by Miranda in [41], S is a choice of section and, following Miranda's terminology, we have a collision of fibers $F_1 + F_2$. In Section I.6.2 we also consider some non-Miranda type collisions.

Chapter I.2

Generalities on genus one fibrations

In this chapter we summarize the basic theory of genus one fibrations.

Definition I.2.0.1. *A genus one fibration is a surjective proper morphism $f : X \to S$ between normal projective varieties, with connected fibers, and such that almost all fibers are smooth curves of genus one. We further assume f is relatively minimal – meaning there are no (-1) curves supported in any fiber.*

Remark I.2.0.2. *When $f : X \to S$ as above admits a (global) section*[1] *we often use the terminology elliptic fibration to reflect the fact that in this case the generic fiber is an elliptic curve over the function field of the base.*

Examples in dimensions two include the product of any two elliptic curves, all surfaces of Kodaira dimension one, Enriques surfaces, Kodaira surfaces, and Dolgachev surfaces. Another beautiful example is the following:

[1]i.e. a morphism $\pi : S \to X$ such that $f \circ \pi = id_S$

Example I.2.0.3. *Consider a pencil of plane cubics $\lambda C + \mu C' = 0$. Any such pencil defines a rational map $\mathbb{P}^2 \to \mathbb{P}^1$ given by $p \mapsto (C(p) : C'(p))$ which is not defined precisely at the nine intersection points $C \cap C'$. Blowing-up $\mathbb{P}^2$ at these nine points resolves the indeterminacy yielding a rational elliptic surface $f : X \to \mathbb{P}^1$ (with section).*

Any genus one fibration has finitely many singular fibers. The possible non-multiple singular fibers have been classified by Kodaira and Néron [32, 33, 45] and Table I.2.1 below gives the full classification. Over a field of characteristic zero, any multiple fiber is of type I_n for some $n \geq 0$ [16, Proposition 5.1.8].

Kodaira Type	**Number of Components**	**Dual Graph**
I_0	1 (smooth)	$\bullet$
I_1	1 (with a node)	$\bullet$
I_n	$n \geq 2$	$\tilde{A}_{n-1}$
II	1 (with a cusp)	$\bullet$
III	2	$\tilde{A}_1$
IV	3	$\tilde{A}_2$
I_n^*	$n+5$	$\tilde{D}_{4+n}$
IV^*	7	$\tilde{E}_6$
III^*	8	$\tilde{E}_7$
II^*	9	$\tilde{E}_8$

Table I.2.1: Kodaira's Classification

Definition I.2.0.4. *Given a genus one fibration $f : X \to S$ we define the **index** of the fibration, and denote it by d_X, as the positive generator of the ideal $\{D \cdot X_\eta \; ; \; D \subset Pic(X)\} \trianglelefteq \mathbb{Z}$, where X_η denotes the generic fiber.*

Note that $d_X = 1$ if and only if f admits a section. Moreover, by assuming that X is projective we have that d_X is always finite.

Remark I.2.0.5. *If K denotes the function field of S, then d_X is the minimal degree of a separable extension L/K such that $X_\eta(L) \neq \emptyset$. We also have that d_X is the index of the image of the restriction map $Pic(X) \to \dfrac{Pic(X_\eta)}{Pic^0(X_\eta)} \simeq \mathbb{Z}$, where the latter isomorphism is given by the degree.*

In the present thesis we are mainly interested in the case when $d_X > 1$, and we will mainly focus in the case $d_X = 2$. In fact, latter in Part II, Chapter II.3 we will further restrict our attention to the situation where X is a rational surface.

I.2.1 The associated Jacobian fibration

One can associate to any genus one fibration without a section, another genus one fibration that has a section – called the associated Jacobian fibration. Below we explain this construction and the main reference we follow is [51, Sections 10.3 and 10.5].

Let $f : X \to S$ be a genus one fibration of index $m > 1$. Then the generic fiber X_η is a smooth genus one curve over the function field of S that has no rational points over this field. Let $Jac(X_\eta)$ denote the corresponding Jacobian variety of divisors of degree 0 on X_η that is, the connected component of the identity of $Pic(X_\eta)$. One can construct an elliptic fibration $J \to S$ with a section whose generic fiber J_η is isomorphic to $Jac(X_\eta)$. This fibration comes with a rational map $\varphi : J \times_S X \to X$ that commutes with the projections to S and has the following properties:

(i) φ is regular on the set of smooth points of fibers of both J and X

(ii) if X_b is a (non-multiple) fiber of X, then the restriction of φ to $J_b^\# \times X_b^\#$ defines a fixed-point-free and transitive action of the group $J_b^\#$ on $X_b^\#$. Here $J_b^\#$ (rep. $X_b^\#$) means the subset of simple points of J_b (resp. X_b) that is, we remove singular points and multiple components.

Note that by construction $X_b^\#$ is a torsor over $J_b^\#$. That is, $X_b^\#$ is a homogeneous space of the group $J_b^\#$ whose elements act without fixed points. In particular, given a point $x \in X_b^\#$ the map $p \mapsto \varphi(p, x)$ defines an isomorphism between $X_b^\#$ and $J_b^\#$, which however depends on the choice of a point x.

Definition I.2.1.1. *The fibration $J \to S$ is called the associated Jacobian fibration (to $f : X \to S$)*[2].

In general, genus one fibrations are classified by their Jacobian fibrations by introducing a group structure in the set of all genus one fibrations with a given Jacobian fibration. Given $f : X \to S$ as above, its class in such group, which we

[2]if Y is a rational surface, then it follows that J is also a rational surface [16, Proposition 5.6.1 (ii)]

denote by $H^1(S, \mathcal{J})$, corresponds to a choice of a closed point $p \in S$ and an element ε_m of order m in $J_P^{\#}$. In fact these ideas can be formalized into a more general result (Lemma I.2.1.4), but in order to state such result we need to first introduce the following definition and some notations.

Definition I.2.1.2. *Given an elliptic curve E over a function field k, we denote by $WC(E/k)$ the Weil-Châtelet group of isomorphism classes of torsors over E which are defined over k.*

Remark I.2.1.3. *The Weil-Châtelet group $WC(E/k)$ can be defined directly from Galois cohomology as $H^1(Gal(\bar{k}/k), E)$, where $\bar{k}$ is an algebraic (resp. separable) closure of k and k has zero (resp. positive) characteristic.*

Given a proper smooth algebraic variety S (over some algebraically closed field) and a closed point $s \in S$, we denote by R_s the strict Henselization[3] of the local ring $\mathcal{O}_{S,s}$ and we define $\eta_s \doteq Spec(k_s)$, where k_s denotes the function field of R_s.

Note that there are natural inclusions $\eta_s = Spec(k_s) \to Spec(R_s) \to S$. So, given any elliptic fibration $J \to S$, we can consider the restriction of J to both $Spec(R_s)$ and η_s. In particular, we define $J_s(\bar{s}) \doteq J \times_\eta \eta_s$, where η is the generic point of S.

Lemma I.2.1.4 ([16, Corollary 5.4.6],[51, p. 207])**.** *If $J \to S$ is an elliptic fibration with a section and with at least one singular fiber, then the map*

$$\tau : H^1(S, \mathcal{J}) \to \bigoplus_{s \in S} WC(J_s(\bar{s})/k_s) = \bigoplus_{s \in S} H^1(Gal(\bar{k_s}/k_s), J_s(\bar{s}))$$

[3] We can take $R_s = \varinjlim_{(U,u)} \mathcal{O}_{U,u}$, where (U, u) runs over all étale neighborhoods of $s \in S$.

is surjective.

Note that for a fixed closed point $s \in S$, we have a natural map

$$\tau_s : H^1(S, \mathcal{J}) \to WC(J_s(\bar{s})/k_s)$$

given by $[X] \mapsto [X_s(\bar{s})]$. Moreover, for a fixed class $[X] \in H^1(S, \mathcal{J})$ we have that $\tau_s([X]) = 0$ for almost all $s \in S$ [16, Corollary 5.4.2]. In fact $\tau_s([X]) = 0$ if and only if the curve $X \times_\eta \eta_s$ has a section if and only if X_s is not multiple.

We call $\tau_s([X])$ the **local invariant** of X at s and we can identify it with an element of finite order (the multiplicity of X_s.) in the Jacobian of X_s.

Remark I.2.1.5 ([16],[51])**.** *The kernel of τ parameterizes fibrations with a section and which are isomorphic to J. Such group is isomorphic to the Brauer group of J and is often referred to in the literature as the Tate-Shafarevich group of J (or of J_η). Note that if J is a rational surface, because the Brauer group is a birational invariant and $Br(\mathbb{P}^2) = 0$, we have that τ is also injective.*

The next lemma will be important latter in Section I.4.2

Lemma I.2.1.6 ([6, Lemma 3.5 and Corollary 3.6])**.** *Given a genus one fibration $f : X \to S$ as above there exists $M \subset X$ a multisection of degree d_X. Moreover, the order of $[X]$ in $H^1(S, \mathcal{J})$ is d_X.*

Chapter I.3

Relative log canonical models

We now recall the basic notions concerning log canonical pairs and we review some basic facts about the log MMP in the relative setting. We state the definitions and results we will use in our computations throughout Sections I.4.1 and I.4.2, and Chapters I.5 and I.6. We refer to [35] and [36] for a more detailed exposition.

Let X be a normal algebraic variety of dimension n and let $\Delta = \sum d_i D_i \subset X$ be a $\mathbb{Q}$-divisor, i.e. a $\mathbb{Q}$-linear combination of prime divisors.

Definition I.3.0.1. *Given any birational morphism $\mu : \tilde{X} \to X$, with $\tilde{X}$ normal, we can write $K_{\tilde{X}} \equiv \mu^*(K_X + \Delta) + \sum a_E E$, where $E \subset \tilde{X}$ are distinct prime divisors, $a_E \doteq a(E, X, \Delta)$ are the discrepancies of E with respect to (X, Δ) and a non-exceptional divisor E appears in the sum if and only if $E = \mu_*^{-1} D_i$ for some i (in that case with coefficient $a(E, X, \Delta) = -d_i$).*

Definition I.3.0.2. *A* ***log resolution*** *of the pair (X, Δ) consists of a proper* *birational morphism* $\mu : \tilde{X} \to X$ *such that* $\tilde{X}$ *is smooth and* $\mu_*^{-1}(\Delta) \cup Exc(\mu)$ *is a simple normal crossings divisor* [1].

Definition I.3.0.3. *We say* (X, Δ) *is* ***log canonical*** *(lc) if* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier and given any log resolution* $\mu : \tilde{X} \to X$ *we have* $K_{\tilde{X}} \equiv \mu^*(K_X + \Delta) + \sum a_E E$ *with all* $a_E \geq -1$*. In particular, if* X *is smooth and* $\Delta = d_i D_i$ *is simple normal crossings, then* (X, Δ) *is log canonical if and only if* $d_i \leq 1$ *for all* i*.*

Definition I.3.0.4. *The number* $lct(X, \Delta) \doteq \sup\{ t \ ; \ (X, t\Delta) \text{ is log canonical}\}$ *is called the* ***log canonical threshold*** *of* (X, Δ)*.*

Definition I.3.0.5. *More generally, given a log canonical pair* (X, Δ) *and a divisor* $D \subset X$*, the number* $lct(X, \Delta, D) \doteq \sup\{t \ ; \ (X, \Delta + tD) \text{ is lc}\}$ *is called the log canonical threshold of* (X, D) *with respect to the pair* (X, Δ)*.*

Remark I.3.0.6. *We can also consider a local version,* $lct_p(X, \Delta)$*, taking the supremum over all* t *such that* $(X, t\Delta)$ *is log canonical in an open neighborhood of* p*, where* $p \in X$ *is a closed point.*

Example I.3.0.7. *When* $X = \mathbb{C}^2$ *and we take as* Δ *a plane curve* $\mathcal{C}$*, then one can easily compute* $lct(\mathbb{C}^2, \mathcal{C})$ *from the Newton Polygon of* $\mathcal{C}$*. For instance, if the Newton Polygon in the* $xy-$*plane contains a vertical edge over the line* $x = x_0$ *and that edge intersects the line* $x = y$*, then* $lct(\mathbb{C}^2, \mathcal{C}) = \dfrac{1}{x_0}$ [38].

[1] that is, each component is smooth and each point étale locally looks like the intersection of $r \leq n$ coordinate hyperplanes

We now observe that given a log pair (X, Δ) (that is, X is a normal variety and $\Delta = \sum d_i D_i$ is a $\mathbb{Q}-$divisor with $0 \leq d_i \leq 1$) and a log resolution $\mu : \tilde{X} \to X$, the discrepancies $a_E = a(E, X, \Delta)$ of any μ-exceptional divisor E satisfy monotonicity:

Lemma I.3.0.8 ([36, Lemma 2.27])**.** *Given (X, Δ) and Δ' effective and $\mathbb{Q}-$Cartier we have that $a(E, X, \Delta + \Delta') \leq a(E, X, \Delta)$.*

Corollary I.3.0.9 ([36, Corollary 2.35(1)])**.** *If $(X, \Delta + \Delta')$ is a log canonical pair and Δ' is an effective and $\mathbb{Q}-$Cartier divisor, then the pair (X, Δ) is also log canonical.*

In fact one can prove the next two results, which give us a way of comparing discrepancies and will be useful latter on.

Lemma I.3.0.10 ([36, Lemma 2.30])**.** *Let $f : \tilde{X} \to X$ be a proper birational morphism between normal varieties. Let $\Delta_{\tilde{X}}$ resp. Δ_X be $\mathbb{Q}-$divisors on $\tilde{X}$ resp. X such that*

$$K_{\tilde{X}} + \Delta_{\tilde{X}} \equiv f^*(K_X + \Delta_X) \qquad \textit{and} \qquad f_* \Delta_{\tilde{X}} = \Delta_X$$

Then for any divisor E over X, $a(E, \tilde{X}, \Delta_{\tilde{X}}) = a(E, X, \Delta_X)$.

Lemma I.3.0.11 ([36, Lemma 3.38])**.** *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "\psi"] \arrow[dr, "\varphi"'] & & X' \arrow[dl, "\varphi'"] \\ & Y & \end{tikzcd}$$

where X, X' and Y are normal varieties, φ and φ' are proper and birational. Let Δ (resp. Δ') be a $\mathbb{Q}-$Cartier divisor on X (resp. on X'). Assume that

1. $\varphi_*\Delta = \varphi'_*\Delta'$

2. $-(K_X + \Delta)$ *is* $\mathbb{Q}-$*Cartier and* $\varphi-$*nef, and*

3. $K_{X'} + \Delta'$ *is* $\mathbb{Q}-$*Cartier and* $\varphi'-$*nef*

Then for any exceptional divisor E *over* Y*,* $a(E, X, \Delta) \leq a(E, X', \Delta')$

I.3.1 The (relative) log MMP

We are now ready to present the notion of **relative log canonical model**, the definition is as follows:

Definition I.3.1.1. *Let* (X, Δ) *be a log canonical pair and* $f : X \to S$ *a proper morphism. A pair* (X^{lc}, Δ^{lc}) *that fits into a diagram as the one below*

$$\begin{tikzcd} (X,\Delta) \arrow[rr, dashed, "\varphi^{lc}"] \arrow[dr, "f"'] & & (X^{lc},\Delta^{lc}) \arrow[dl, "f^{lc}"] \\ & S & \end{tikzcd}$$

is called a log canonical model of (X, Δ) *over* S *(or relative with respect to* f*) if*

(i) f^{lc} *is proper*

(ii) $(\varphi^{lc})^{-1}$ *has no exceptional divisors*

(iii) $\Delta^{lc} = \varphi^{lc}_*\Delta$

(iv) $K_{X^{lc}} + \Delta^{lc}$ *is* $f^{lc}-$*ample and*

(v) $a(E, X, \Delta) \leq a(E, X^{lc}, \Delta^{lc})$ *for every* $\varphi^{lc}-$*exceptional divisor* $E \subset X$

A natural question then is whether such objects exist and (if they do) whether they are unique.

Theorem I.3.1.2 ([36, Theorem 3.52])**.** *Let (X, Δ) be a log canonical pair and let $f : X \to S$ be a proper morphism. If it exists, a log canonical model (X^{lc}, Δ^{lc}) is unique and*

$$X^{lc} = Proj_S \left(\bigoplus_{m \geq 0} f_* \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor) \right)$$

Existence is given by the (relative) log MMP, which takes as an input a log canonical pair (X, Δ) and a projective morphism $f : X \to S$ and applies Theorem I.3.1.3 below several times in order to get a birational model $(f' : X' \to S, \Delta')$, with $K'_X + \Delta'$ an $f'-$nef divisor (see also Remark I.3.1.4). Abundance (which holds in dimensions 2 and 3, see e.g. [14] and [48]) then implies $K'_X + \Delta'$ is $f'-$semiample and the image of (X', Δ') under the corresponding morphism to some $\mathbb{P}^N$ is the relative log canonical model. Note that such morphism contracts precisely those curves C for which $(K'_X + \Delta') \cdot C = 0$.

Theorem I.3.1.3 ([36, Theorem 3.25])**.** *Let (X, Δ) be a log canonical pair and let $f : X \to S$ be a projective morphism. Assume dim $X = 2$ or 3*[2]*. Then*

(i) There exist countably many rational curves $C_j \subset X$ contracted by f and such that

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{(K_X + \Delta) \geq 0} + \sum_j \mathbb{R}_{\geq 0}[C_j]$$

[2]Theorem I.3.1.3 holds for arbitrary dimensions, but to our purposes we only need the log MMP for surface and threefold pairs.

with $0 < -(K_X + \Delta) \cdot C_j \leq 2 \; dim X$ *and such that* $R_j \doteq \mathbb{R}_{\geq 0}[C_j]$ *is an extremal ray for each* j.

(ii) Given any extremal ray R*, there exists a unique morphism* $\varphi_R : X/S \to Y/S$ *such that* $(\varphi_R)_* \mathcal{O}_X = \mathcal{O}_Y$ *and an irreducible curve* $C \subset X$ *is contracted by* φ_R *if and only if* $[C] \in R$*. The morphism* φ_R *is called an extremal contraction.*

Above, $\overline{NE}(X/S)$ denotes the Mori cone of X relative to f. That is, the closure of the convex cone $N_1(X) \doteq (Z_1(X)/\equiv) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by those classes of effective irreducible curves (1-cycles) which are contracted by f. Further, we say a ray R of such cone is an extremal ray if it satisfies the following condition: if $x, y \in \overline{NE}(X/S)$ are such that $x + y \in R$, then $x, y \in R$.

Remark I.3.1.4. *In Theorem I.3.1.3, each extremal contraction* φ_R *which is birational is either a divisorial contraction or a small contraction. In particular, existence and termination of flips must also hold for a (relative) log canonical model to exist.*

More generally, we can define the relative log canonical model of a log pair by considering a log resolution:

Definition I.3.1.5. *Given a log pair* (X, Δ) *and a proper morphism* $X \to S$*, its relative log canonical model is the relative log canonical model of* $(\tilde{X}, \mu_*^{-1}(\Delta) + Exc(\mu))$*, where* $\mu : \tilde{X} \to X$ *is any log resolution.*

Chapter I.4

Relative log canonical models for genus one fibrations in dimension two

In this chapter we run the relative log MMP for genus one fibrations in dimension two. More precisely, in Section I.4.1 we consider elliptic surfaces and the boundary divisor Δ we fix is supported in a section plus a weighted fiber. But different from [2] we don't take the fiber to be reduced. Section I.4.2 is then dedicated to running the relative log MMP for explicit examples of elliptic surfaces pairs, where Δ is supported in a multisection plus a weighted fiber.

I.4.1 Elliptic surfaces with section and the Weierstrass model

Let $f : X \to C$ be a relatively minimal elliptic surface with a section S. For any choice of fiber F and a weight $0 \leq a \leq 1$, we will refer to the pair $(X, S + aF)$ as an **elliptic surface pair**. Our goal in this Section is to run the relative log MMP for elliptic surface pairs.

Contracting all the (finitely many) fiber components not meeting S yields so called (minimal) Weierstrass model $f' : W \to C$. and we will write $(W, S' + aF')$ for the corresponding pair in the Weierstrass model. That is, $F' \doteq \varphi_* F$ and $S' \doteq \varphi_* S$, where $\varphi : X \to W$ is the birational map defining W.

Now, because $\varphi : X \to W$ is a minimal resolution which is also crepant (since W has only canonical singularities[1]), we can prove:

Proposition I.4.1.1. *Given an elliptic surface pair $(X, S + aF)$ as above and a choice of weight $0 \leq a \leq lct(X, F)$, its relative log canonical model is the minimal Weierstrass model independent of the type of the fiber F.*

Proof. Note that the choice of the weight is such that $(X, S + aF)$ is log canonical. Now, we know that $\varphi : X \to W$ is a minimal crepant resolution, hence

$$K_X + S + aF = \varphi^*(K_W + S' + aF')$$

[1] in fact only rational double points

where $S' + aF' = \varphi_*(S + aF)$. In particular, by Lemma I.3.0.10, we have

$$a(E, X, S + aF) = a(E, W, S' + aF')$$

for any $\varphi-$exceptional divisor. But then, the pair $(W, S' + aF')$ satisfies Definition I.3.1.1, since $(K_W + S' + aF') \cdot \gamma = 1 > 0$ for any irreducible curve γ supported on a fiber of $f' : W \to C$. That is, $K_W + S' + aF'$ is $f'-$ample. □

Note that when F is of type I_n, II, III or IV, then F is reduced. The case where the marked divisor F is taken to be reduced was studied in [2]. Proposition I.6.1.1 above gives us an intrinsic way of partially recovering their result for $0 \leq a \leq lct(X, F)$.

I.4.2 Elliptic surfaces with multisections

Given a projective surface X together with a genus one fibration $f : X \to C$ as in Definition I.2.0.1 and a choice of multisection M of degree $m > 1$, let us assume M intersects some fixed singular fiber F transversally. Taking the fiber F to be reduced [2] and with some weight $0 \leq a \leq 1$, in this section we will still refer to the pair $(X, M + aF)$ as an elliptic surface pair.

One of the goals of this section is then to compute the relative log canonical model for several examples of pairs $(X, M + aF)$ as above as an illustration of the general statements of Propositions I.4.2.10 and I.4.2.13.

[2]meaning we consider the reduced divisor associated to some geometric fiber, i.e. $F = f^{-1}(p)_{red}$ for some closed point $p \in C$

First, in order to fix notations, note that whenever the pair $(X, M + aF)$ is not log canonical we need to first take a log resolution

$$\varphi : (Z, \tilde{M} + a\tilde{F} + \mathrm{Exc}(\varphi)) \to (X, M + aF)$$

We will write $\tilde{F}$ (resp. $\tilde{M}$) to denote the strict transform (under φ) of F (resp. M) and we will mark the exceptional divisor $\mathrm{Exc}(\varphi)$ with coefficient one. The relative lc model of $(X, M + aF)$ is, by definition, the relative lc model of $(Z, \tilde{L} + a\tilde{F} + \mathrm{Exc}(\varphi))$ (see Definition I.3.1.5).

Note also that relative log canonical model of a pair $(X, M+aF)$ always contracts any irreducible fiber component that is not supported on F and which does not intersect the multisection M. Therefore, in what follows, we will only describe the boundary divisor of the relative lc model.

Definition I.4.2.1. *Given an elliptic surface pair $(X, M + aF)$ consider its relative log canonical model $\varphi^{lc} : (X, M + aF) \to (X^{lc}, M^{lc} + F_a^{lc})$. We say $(X^{lc}, M^{lc} + F_a^{lc})$ is a **twisted model** if F_a^{lc} is supported in a non-reduced divisor E^{lc}. We call it an **intermediate model** if F_a^{lc} is supported in a normal-crossings union of divisors $A^{lc} + E^{lc}$, where A^{lc} consists of the fiber components meeting the multisection M.*

We observe that the following two results hold in general.

Proposition I.4.2.2. *Consider an elliptic surface pair $(X, M + aF)$, with F a fiber of type I_n, II, III or IV and $0 \leq a \leq lct(X, M, F)$. Then the relative log canonical model contracts every irreducible fiber component not meeting the multisection M.*

The same is true if we replace M by a "weighted multisection" $a_M M$, for some weight $0 < a_M \leq 1$.

In fact if we consider F possibly non-reduced[3], then we have the following more general statement, which is independent of the type of F:

Proposition I.4.2.3. *Consider an elliptic surface pair $(X, M + aF)$, with F now a fiber which we take not necessarily reduced and let $0 \leq a \leq lct(X, M, F)$. Then the relative log canonical model contracts every irreducible fiber component not meeting the multisection M. Again, the same is true if we replace M by a "weighted multisection" $a_M M$, where $0 < a_M \leq 1$ is a choice of weight.*

Proof. If $0 \leq a \leq lct(X, M, F)$, then the pair $(X, a_M M + aF)$ is log canonical and we have that $(K_X + a_M M + aF) \cdot \gamma \geq a_M > 0$ if γ meets the multisection and $(K_X + a_M M + aF_1 + bF_2) \cdot \gamma = 0$ otherwise. In particular, $K_X + a_M M + aF$ is already $f-$nef, hence $f-$semiample by abundance, and the relative log canonical model contracts precisely the irreducible fiber components not meeting M. □

Remark I.4.2.4. *Note that the proof above also includes Proposition I.4.2.2 since those types of fibers are already reduced.*

Corollary I.4.2.5. *Consider an elliptic surface pair $(X, M + aF)$ with F a fiber of type I_n, II, III or IV and $0 \leq a \leq lct(X, M, F)$. If the degree of M is greater or equal than the number of irreducible components of F, then the relative log canonical*

[3]meaning $F = f^{-1}(p)$ for some closed point $p \in C$, i.e., F is really a geometric fiber

model is the pair $(X, M + aF)$ *itself. Again, one can replace* M *by* $a_M M$*, for some* $0 < a_M \leq 1$ *fixed.*

Remark I.4.2.6. *An analogous statement holds for the non-reduced case.*

Remark I.4.2.7. *If* $M \cap F$ *is supported in the smooth locus of* F *and* M *intersects* F *transversally, then* $lct(X, M, F) = lct(X, F)$*.*

I.4.2.1 Elliptic K3 surfaces

A K3 surface is a projective smooth variety X of dimension 2 such that $\omega_X \simeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

Examples.

(i) a smooth quartic surface $X \subset \mathbb{P}^3$

By adjunction, $\omega_X = \omega_{\mathbb{P}^3} \otimes \mathcal{O}(4)\big|_X \simeq \mathcal{O}_X$*. Moreover, the short exact sequence*

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

of invertible sheaves on $\mathbb{P}^3$ *induces a long exact in cohomology so that the vanishing of* $H^1(\mathbb{P}^3, \mathcal{O})$ *and* $H^2(\mathbb{P}^3, \mathcal{O}(-4))$ *implies that* $H^1(X, \mathcal{O}_X) = 0$*.*

(ii) a divisor of bidegree $(2, 3)$ *in* $\mathbb{P}^1 \times \mathbb{P}^2$

Again, adjunction gives us $\omega_X \simeq \mathcal{O}_X$ *and the vanishing* $H^1(X, \mathcal{O}_X) = 0$ *follows from the long exact sequence in cohomology that is induced from the ideal sheaf sequence in* $\mathbb{P}^1 \times \mathbb{P}^2$ *for the divisor of bidegree* $(2, 3)$*.*

(iii) a degree 2 cover $\pi : X \to \mathbb{P}^2$ branched along a curve of degree 6

If we apply the canonical bundle formula for branched covers to $\pi : X \to \mathbb{P}^2$ we get that $\omega_X = \pi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{O}(3)) \simeq \mathcal{O}_X$. Moreover, $\pi_* \mathcal{O}_X \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}(-3)$ together with the projection formula (for cohomology) give us $H^1(X, \mathcal{O}_X) = 0$.

It is a well known fact (see e.g. [26]) that a K3 surface X admits an elliptic fibration if and only if $\exists L \in NS(X)$ such that $L^2 = 0$. Moreover, if that is the case, then the base curve has to be rational, i.e., $\simeq \mathbb{P}^1$. Further, any $X \to \mathbb{P}^1$ elliptic K3 is relatively minimal and does not have multiple fibers.

Other classical invariants are encoded in the Hodge diamond of a K3 surface:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

I.4.2.1.1 Quartics in $\mathbb{P}^3$ containing a line

We will consider an example of an elliptic K3 surface $X \to \mathbb{P}^1$ with a multisection $M \subset X$ of degree 3 and two fibers of type IV. In Lemma I.4.2.8 we will fix F to be one of these fibers and we will compute the relative log canonical model of $(X, M + aF)$ for $0 \leq a \leq 1$.

Let $X \subset \mathbb{P}^3$ be a smooth quartic, then X is a K3 surface. Let us assume that

X contains a line L. Let $|D| = \{\text{planes in } \mathbb{P}^3 \text{ containing } L\}$. For $H \in |D|$ define $E \doteq H - L$, then $E^2 = 0$ and therefore X admits and elliptic fibration. Moreover, L gives us a multisection of degree 3.

Explicitly [39, p. 235], let $X = X(q_1, q_2)$ be given by $q_1(x, y) = q_2(z, w)$, where

$$\begin{aligned} q_1(x,y) &= xy(x-y)(x-\lambda y) \\ q_2(z,w) &= zw(z-w)(z-\mu w) \end{aligned}$$

and $\lambda, \mu \in \mathbb{C}$.

Consider the following two elliptic curves: $E_1 : y^2 = q_1(x, 1)$ and $E_2 : y^2 = q_2(z, 1)$.

If E_1 and E_2 have different $j-$invariants, then X contains exactly 16 lines (and assuming $\lambda, \mu \neq 1$) [39, Proposition 1.4]:

$$\ell_1 : \begin{cases} x = 0 \\ z = 0 \end{cases} \quad \ell_2 : \begin{cases} x = 0 \\ w = 0 \end{cases} \quad \ell_3 : \begin{cases} x = 0 \\ z = w \end{cases} \quad \ell_4 : \begin{cases} x = 0 \\ z = \mu w \end{cases}$$

$$\ell_5 : \begin{cases} y = 0 \\ z = 0 \end{cases} \quad \ell_6 : \begin{cases} y = 0 \\ w = 0 \end{cases} \quad \ell_7 : \begin{cases} y = 0 \\ z = w \end{cases} \quad \ell_8 : \begin{cases} y = 0 \\ z = \mu w \end{cases}$$

$$\ell_9 : \begin{cases} x = y \\ z = 0 \end{cases} \quad \ell_{10} : \begin{cases} x = y \\ w = 0 \end{cases} \quad \ell_{11} : \begin{cases} x = y \\ z = w \end{cases} \quad \ell_{12} : \begin{cases} x = y \\ z = \mu w \end{cases}$$

$$\ell_{13} : \begin{cases} x = \lambda y \\ z = 0 \end{cases} \quad \ell_{14} : \begin{cases} x = \lambda y \\ w = 0 \end{cases} \quad \ell_{15} : \begin{cases} x = \lambda y \\ z = w \end{cases} \quad \ell_{16} : \begin{cases} x = \lambda y \\ z = \mu w \end{cases}$$

Now, choose $L = \ell_1$ and consider the plane $H : z = tx$ so that we get an elliptic fibration

$$\begin{aligned} f : X &\to \mathbb{P}^1 \\ [x : y : z : w] &\mapsto [x : z] \end{aligned}$$

with fiber the plane cubic

$$y(x-y)(x-\lambda y) = tw(tx-w)(tx-\mu w)$$

Note that $f^{-1}(0) = \{\ell_5, \ell_9, \ell_{13}\}$ meeting at the point $[1:0:0:0]$ that is, we have a type IV fiber. Similarly, $f^{-1}(\infty) = \{\ell_2, \ell_3, \ell_4\}$ meeting at the point $[0:0:0:1]$ and again we have a type IV fiber.

Moreover, we can also argue that no other singular fiber can contain a line because all other lines except $L = \ell_1$ intersect only one of ℓ_2, ℓ_3, ℓ_4 (and only one of $\ell_5, \ell_9, \ell_{13}$) that is, they define sections.

In particular, we can only have type I_1 or type II as possibilities for the other singular fibers. We observe that this agrees with the classification given by Shimada in [53]. We thus have the following nine possible configurations:

$2IV + 8II$	$2IV + 7II + 2I_1$	$2IV + 6II + 4I_1$
$2IV + 5II + 6I_1$	$2IV + 4II + 8I_1$	$2IV + 3II + 10I_1$
$2IV + 2II + 12I_1$	$2IV + II + 14I_1$	$2IV + 16I_1$

We can even say what the Picard number $\rho(X)$ is for such elliptic surface. Theorem 1.3 in [39] tells us that $\rho(X)$ only depends on the curves E_1 and E_2:

(i) $\rho(X) = 18$ if E_1 and E_2 are not isogenous,

(ii) $\rho(X) = 19$ if E_1 and E_2 are isogenous and do not have complex multiplication and

(iii) $\rho(X) = 20$ if E_1 and E_2 are isogenous and do have complex multiplication.

In particular, by the Shioda-Tate formula, we also know what the rank ($\doteq r$) of the Mordell-Weil group is [39, Theorem 1.5] :

(i) $r = 12$ if E_1 and E_2 are not isogenous,

(ii) $r = 13$ if E_1 and E_2 are isogenous and do not have complex multiplication and

(iii) $r = 14$ if E_1 and E_2 are isogenous and do have complex multiplication.

Next, we observe that $\ell_1 \cap \ell_2 = \ell_1 \cap \ell_3 = \ell_1 \cap \ell_4 = \{[0:1:0:0]\}$. That is, the multisection $L = \ell_1$ meets the type IV fiber over $t = \infty$ at the triple point.

Similarly, $\ell_1 \cap \ell_5 = \ell_1 \cap \ell_9 = \ell_1 \cap \ell_{13} = \{[0:0:0:1]\}$.

In fact, the Riemann-Hurwitz formula applied to the degree 3 cover $L \to \mathbb{P}^1$ gives us that the two type IV fibers are the only ones which are ramified. As a consequence, combining Corollary I.4.2.5 and Lemma I.4.2.8 we can completely characterize the relative log canonical model of the pair $\left(X, L + \sum a_i F_i\right)$, where $0 \leq a_i \leq 1$ and F_i are all the singular fibers of the fibration $X \to \mathbb{P}^1$ constructed in this example.

If we fix just one of the two type IV fibers we obtain the following:

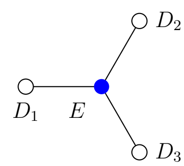
Lemma I.4.2.8. *Consider $f : X = X(q_1, q_2) \to \mathbb{P}^1$ as in the above example and fix F one of the two type IV fibers. If $\varphi : (Z, \tilde{L} + a\tilde{F} + Exc(\varphi)) \to (X, L + aF)$ is a log resolution, then the relative log canonical model is:*

(i) the pair $(X, L + aF)$ itself for all $0 \leq a \leq 1/3$ (See also Corollary I.4.2.5)

(ii) the log resolution for all $1/3 < a < 2/3$ that is, the pair $(Z, \tilde{M} + a\tilde{F} + Exc(\varphi))$

(iii) a twisted model for all $2/3 \leq a \leq 1$ and the log canonical model contracts $\tilde{F}$.

Proof. The pair $(X, L + aF)$ is not normal crossings, so before running the log MMP we consider $\varphi : (Z, \tilde{L} + a\tilde{F} + \mathrm{Exc}(\varphi)) \to (X, L + aF)$ a log resolution, where $\tilde{F}$ (resp. $\tilde{L}$) denotes the strict transform of F (resp. L). The dual graph of the corresponding fiber on the log resolution is given by



where the component meeting the multisection is marked by the blue node.

The log resolution $\varphi : Z \to X$ is obtained after blowing-up the singular point of F, hence we get only one exceptional divisor E with self-intersection -1 and we have $\tilde{F} = D_1 + D_2 + D_3$, which are all -3 curves. Moreover, $K_Z = \varphi^* K_X + E$, so that $K_Z \cdot D_i = 1$ and $K_Z \cdot E = -1$.

We can now run the log MMP. First, we compute $(K_Z + \tilde{L} + a\tilde{F} + E) \cdot \gamma$ for any irreducible curve γ supported in $\varphi^{-1}(F)$.

We find:

$$(K_Z + \tilde{L} + a\tilde{F} + E) \cdot D_i = 2 - 3a$$

$$(K_Z + \tilde{L} + a\tilde{F} + E) \cdot E = 3a - 1$$

If $1/3 < a < 2/3$, then the above tell us $\Delta \doteq (K_Z + \tilde{L} + a\tilde{F} + E)$ is already $f \circ \varphi-$ample. If $a = 2/3$, then such divisor is $f \circ \varphi-$nef hence, by abundance, the log canonical model contracts all curves γ such that $\Delta \cdot \gamma = 0$. Those are precisely the D_i.

Now, if $2/3 < a \leq 1$, then there exists a morphism $\mu : Z \to Z'$ contracting all the D_i. Writing $D' \doteq \mu_* D$ for any divisor D in Z it follows that $\mu^* E' = E + 1/3D_1 + 1/3D_2 + 1/3D_3$. In particular, $E' \cdot E' = 0$ and $K'_Z \cdot E' = 0$, by the projection formula. As a consequence, $(K_{Z'} + L' + aF' + E') \cdot E' = L' \cdot E' = 1 > 0$, where $L' \doteq \mu_* \tilde{L}$ and, similarly, $F' \doteq \mu_* \tilde{F}$.

But then Δ' is $f'-$ample, where $f' : Z' \to C$ is the associated fibration, and the log canonical model is the twisted model.

Finally, if $0 \leq a < 1/3$, then there exists a morphism $\varepsilon : Z \to Z''$ contracting E which is precisely the blow-up and if $a = 1/3$, then Δ is already $f \circ \varphi-$nef hence, by abundance, E gets contracted as well. $\square$

Remark I.4.2.9. *Note that in this example the 3-section L intersects the singular locus of F. In particular, $1/3 = lct(X, L, F) \neq lct(X, F) = 2/3$.*

Note that Lemma I.4.2.8 also applies to any elliptic surface pair $(X, M + aF)$, where M is a multisection of degree 3 intersecting a fiber F of type IV at the triple point. The proof above does not depend on a description of X. We have:

Proposition I.4.2.10. *Let $(X, M + aF)$ be an elliptic surface pair where M is a* *multisection of degree* 3 *intersecting a fiber* F *of type* IV *at its triple point. If*

$$\varphi : (Z, \tilde{M} + a\tilde{F} + Exc(\varphi)) \to (X, M + aF)$$

is a log resolution, then the relative log canonical model is:

(i) the pair $(X, M + aF)$ *itself for all* $0 \leq a \leq 1/3$ *(See also Corollary I.4.2.5)*

(ii) the log resolution for all $1/3 < a < 2/3$ *that is, the pair* $(Z, \tilde{M} + a\tilde{F} + Exc(\varphi))$

(iii) a twisted model for all $2/3 \leq a \leq 1$ *and the log canonical model contracts* $\tilde{F}$*.*

Perhaps the main interesting feature of the above result lies in the following observation: If we compare it to the classification in [2] for elliptic surface pairs with a marked section, then we have replaced the Weierstrass model with the pair $(X, M + aF)$ itself since M meets all three components of F. Further, we have replaced the "intermediate model" by the log resolution $(Z, \tilde{M} + a\tilde{F} + E)$.

In addition, note that we have $lct(X, M, F) = 1/3$ so that (i) is a particular case of the more general statement of Corollary I.4.2.5.

A similar picture also appears in the next example we consider.

I.4.2.1.2 Surfaces of bidegree $(2, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$

In this next example we consider an elliptic K3 surface $X \to \mathbb{P}^1$ with a multisection M of degree 3 and that, generically, has six singular fibers of type IV and does not admit a section. In Lemma I.4.2.11 we will fix F to be one of these fibers and we will compute the relative log canonical model of $(X, M + aF)$ for $0 \leq a \leq 1$.

Let $X \subset \mathbb{P}^1 \times \mathbb{P}^2$ be a divisor of bidegree $(2,3)$. If $p_1 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ and $p_2 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ are the standard projections, then the divisors $D_1 \doteq p_1^* \mathcal{O}_{\mathbb{P}^1}(1)$ and $D_2 \doteq p_2^* \mathcal{O}_{\mathbb{P}^2}(1)$ generate $NS(X)$ and satisfy $D_1^2 = 0, D_2^2 = 2$ and $D_1 \cdot D_2 = 3$. That is, D_1 represents a fiber and D_2 a multisection of degree 3.

Explicitly, consider the surface X defined by $aX^3 + bY^3 + cZ^3 = 0$, where $[X : Y : Z]$ are coordinates on $\mathbb{P}^2$ and a, b, c are homogeneous polynomials of degree 2. We call such surface of Fermat type and if a, b and c are generic, then X has six singular fibers of type IV.

By the discussion above, there are at least three multisections of degree 3. Namely, the ones given by $X = 0, Y = 0$ and $Z = 0$. Moreover, the points where each 3-section meets a singular fiber are inflection points of the cubic in $\mathbb{P}^2$, hence 3-torsion points. There are 9 of them.

For $f : X \to \mathbb{P}^1$ of Fermat type we fix a fiber F of type IV and M a 3-section. We then have the following:

Lemma I.4.2.11. *If $\varphi : (Z, \tilde{M} + a\tilde{F} + Exc(\varphi)) \to (X, M + aF)$ is a log resolution, then the relative log canonical model of the pair $(X, M + aF)$ is:*

(i) the pair $(X, M + aF)$ itself for $0 \leq a \leq lct(X, F)$ (See also Corollary I.4.2.5)

(ii) the log resolution for all $lct(X, F) < a < 1$

(iii) a twisted model for $a = 1$ and the log canonical model contracts $\tilde{F}$.

Proof. The multisection M intersects F at all three components. Moreover, such intersection $M \cap F$ is transversal and supported in the smooth locus of F. As a consequence, one can check that the pair $(X, M + aF)$ is log canonical if and only if $0 \leq a \leq lct(X, F)$.

Now, if γ is an irreducible curve supported on F, then

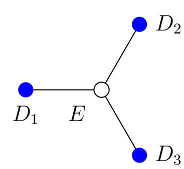
$$(K_X + M + aF) \cdot \gamma = 1 > 0$$

that is, the divisor $(K_X + M + aF)$ is $f-$ample. This proves (i).

If $2/3 = lct(X, F) < a$, then we need to consider

$$\varphi : (Z, \tilde{M} + a\tilde{F} + \mathrm{Exc}(\varphi)) \to (X, M + aF)$$

a log resolution, where $\tilde{F}$ (resp. $\tilde{M}$) denotes the strict transform of F (resp. M). Below we represent the dual graph of the corresponding fiber



where the components meeting the multisection are marked by the blue nodes.

The log resolution $\varphi : Z \to X$ is obtained after a unique blow-up of the singular point of F, so that we have only one exceptional divisor E with self-intersection -1 and $\tilde{F} = D_1 + D_2 + D_3$, which are all -3 curves. Moreover, $K_Z = \varphi^* K_X + E$, so that $K_Z \cdot D_i = 1$ and $K_Z \cdot E = -1$.

Next, we run the log MMP. We compute $(K_Z + \tilde{M} + a\tilde{F} + E) \cdot \gamma$ for any irreducible curve γ supported in $\varphi^{-1}(F)$.

We find:

$$(K_Z + \tilde{M} + a\tilde{F} + E) \cdot D_i = 3 - 3a$$

$$(K_Z + \tilde{M} + a\tilde{F} + E) \cdot E = 3a - 2$$

In particular, if $2/3 < a < 1$, then the divisor $\Delta \doteq K_Z+\tilde{M}+a\tilde{F}+E$ is $f\circ\varphi-$ample and the log canonical model is the log resolution, i.e., the pair $(Z, \tilde{M}+a\tilde{F}+\mathrm{Exc}(\varphi))$. If $a = 1$, then Δ is $f\circ\varphi-$nef and, by abundance, the log canonical model contracts all the D_i, yielding the "twisted model". □

Remark I.4.2.12. *We note that the example above was considered in* [28] *for constructing an elliptic Calabi-Yau* $4-$*fold without section as a product of two K3 surfaces, where one is taken to be of Fermat type.*

Again, Lemma I.4.2.11 applies to any elliptic surface pair $(X, M + aF)$, where M is a multisection of degree 3 intersecting a fiber F of type IV at all the three components in smooth points. We have:

Proposition I.4.2.13. *Let* $(X, M + aF)$ *be an elliptic surface pair where* M *is a multisection of degree* 3 *intersecting a fiber* F *of type* IV *at all the three components in smooth points. If* $\varphi : (Z, \tilde{M} + a\tilde{F} + Exc(\varphi)) \to (X, M + aF)$ *is a log resolution, then the relative log canonical model is:*

(i) the pair $(X, M + aF)$ *itself for* $0 \le a \le lct(X, F) = 2/3$ *(See also Corollary I.4.2.5)*

(ii) the log resolution for all $lct(X, F) < a < 1$ *that is, the pair* $(Z, \tilde{M}+a\tilde{F}+Exc(\varphi))$

(iii) a twisted model for $a = 1$ and the log canonical model contracts $\tilde{F}$.

I.4.2.1.3 Double covers of $\mathbb{P}^2$ branched along a sextic

The last two examples of elliptic K3 surfaces we consider are given by double covers $X \to \mathbb{P}^2$ branched along six lines in general position.

We will construct examples with multisections M of degree 2 and of degree 3. Moreover, after fixing a singular and reduced fiber F, we will compute the relative log canonical model of $(X, M + aF)$ for $0 \leq a \leq 1$. Their classification is the content of Lemmas I.4.2.17 through I.4.2.20 below. Although we are making statements referring to the explicit examples, these statements hold with generality. That is, our classification only depends on the type of the marked singular fiber and how the fixed multisection intersects it, but not on the surface itself.

Consider $L_i \subset \mathbb{P}^2$, with $i = 1, \ldots, 6$, six lines in general position, i.e., no three of the lines are concurrent. Let $P_{i,j}$ for $i < j$ denote the 15 intersection points determined by such lines. That is, $P_{i,j} \doteq L_i \cap L_j$ $(i < j)$. Then there exists a double cover $\varphi : Y \to \mathbb{P}^2$ whose branching divisor consists precisely of the lines L_i and we can construct a K3 surface X by resolving the 15 double points of Y. Moreover, the rational map $X \dashrightarrow \mathbb{P}^2$ factors through $X \to X/\langle\sigma\rangle \simeq \tilde{P}$, where σ is the induced involution on X and $\tilde{P}$ is the blow-up of $\mathbb{P}^2$ at the points $P_{i,j}$. Further, explicit choices of a base-point-free linear system $|D|$ with $D^2 = 0$ give elliptic fibrations $X \to \mathbb{P}^1$.

Let $Q_{i,j} \doteq \varphi^* P_{i,j}$ and define $l_{i,j} \subset X$ to be the exceptional divisor over the double point $Q_{i,j}$ (for $i < j$). Let l_i be the rational curve so that $2l_i$ is the strict transform of $\varphi^* L_i$. Then

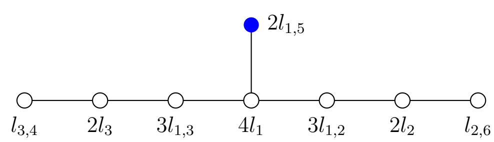
Lemma I.4.2.14 ([31, Lemma 5.2])**.**

$$l_i \cdot l_j = \delta_{i,j} \qquad l_{i,j} \cdot l_{k,m} = -2\delta_{i,k}\delta_{j,m} \qquad l_i \cdot l_{k,m} = \delta_{i,k} + \delta_{i,m}$$

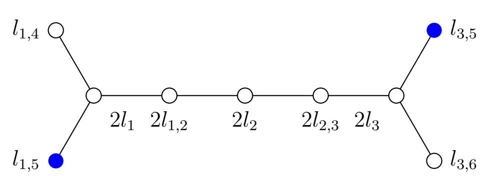
In the examples below we construct elliptic K3 surfaces $X \to \mathbb{P}^1$ with multisections by conveniently choosing D as some linear combination of the rational curves l_i and $l_{i,j}$:

Example I.4.2.15 ([31])**.**

(i) *Choose* $D = l_{3,4} + 2l_3 + 3l_{1,3} + 2l_{1,5} + 4l_1 + 3l_{1,2} + 2l_2 + l_{2,6}$. *Then* D *corresponds to a type* III^* *fiber and* l_5 *is a multisection of degree 2 for* $X \to \mathbb{P}^1$. *Moreover, such multisection intersects the type* III^* *fiber as indicated by the blue node in the graph below:*

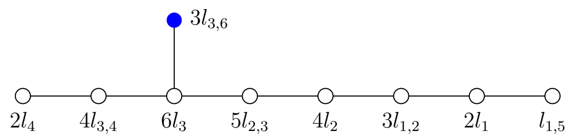


(ii) *By choosing* $D = l_{1,5} + l_{1,4} + 2l_1 + 2l_{1,2} + 2l_2 + 2l_{2,3} + 2l_3 + l_{3,5} + l_{3,6}$ *we get a fiber of type* I_4^* *so that the corresponding fibration* $X \to \mathbb{P}^1$ *has* l_5 *as a two-section, intersecting the* I_4^* *fiber as indicated by the blue nodes in the graph below:*

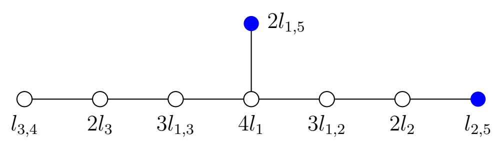


Example I.4.2.16 ([31])**.**

(i) *Take* $D = l_{1,5}+2l_1+3l_{1,2}+4l_2+5l_{2,3}+6l_3+4l_{3,4}+2l_4+3l_{3,6}$. *Then* D *corresponds to a fiber of type* II^* *and* $X \to \mathbb{P}^1$ *has* l_6 *as a trisection, intersecting the type* II^* *fiber as indicated in the graph below by the blue node:*



(ii) *If we choose* $D = l_{3,4}+2l_3+3l_{1,3}+4l_1+2l_{1,5}+3l_{1,2}+2l_2+l_{2,5}$, *then* D *corresponds to a fiber o type* III^* *and the rational curve* l_5 *is a trisection intersecting such fiber as indicated by the blue nodes in the graph below:*



Lemma I.4.2.17. *Let* $X \to \mathbb{P}^1$ *be the elliptic K3 surface constructed in Example I.4.2.15 (i). Write* $M = l_5$ *and let* F *be the reduced divisor associated to the fiber of type* III^*. *Then the relative log canonical model of the pair* $(X, M + aF)$

1. *contracts every irreducible fiber component not meeting* M *for* $a = 0$
2. *is an intermediate model for all* $0 < a < 1$
3. *is a twisted model for* $a = 1$

Proof. With the notations from Example I.4.2.15 (i) we compute:

$$\begin{aligned}
(K_X + \Delta) \cdot l_{3,4} &= -a \\
(K_X + \Delta) \cdot l_{2,6} &= -a \\
(K_X + \Delta) \cdot l_2 &= 0 \\
(K_X + \Delta) \cdot l_3 &= 0 \\
(K_X + \Delta) \cdot l_{1,3} &= 0 \\
(K_X + \Delta) \cdot l_{1,2} &= 0 \\
(K_X + \Delta) \cdot l_1 &= a \\
(K_X + \Delta) \cdot l_{1,5} &= 1 - a
\end{aligned}$$

where $\Delta \doteq M + aF$. In particular, for $a = 0$ we see that the log canonical model contracts every irreducible fiber component not meeting M. If $a > 0$ we conclude that there exists a morphism $\mu : X \to X_1$ contracting the curves $l_{3,4}$ and $l_{2,6}$. Using the projection formula we find that

$$\begin{aligned}
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_2 &= -a/2 \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_3 &= -a/2 \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_{1,3} &= 0 \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_{1,2} &= 0 \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_1 &= a \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_{1,5} &= 1 - a
\end{aligned}$$

and we see that there exists a morphism $X_1 \to X_2$ further contracting the curves l_2

and l_3. By computing the relevant intersection numbers (as above) one finds that that there exists a third morphism $X_2 \to X_3$ contracting the curves $l_{1,3}$ and $l_{1,2}$. If $\psi : X \to X_3$ denotes the composition $X \to X_1 \to X_2 \to X_3$ of these three morphisms, then

$$\begin{aligned} (K_{X_3} + \psi_*\Delta) \cdot \psi_* l_1 &= a - a/4 - a/4 = a/2 > 0 \\ (K_{X_3} + \psi_*\Delta) \cdot \psi_* l_{1,5} &= 1 - a \end{aligned}$$

which finally tells us the log canonical model is an intermediate model for all $0 < a < 1$ and a twisted model for $a = 1$. Moreover, with the notations introduced in Definition I.4.2.1, we have that $A^{lc} = \varphi_*^{lc} l_{1,5}$ and $E^{lc} = \varphi_*^{lc} l_1$. □

Lemma I.4.2.18. *Let $X \to \mathbb{P}^1$ be the elliptic K3 surface constructed in Example I.4.2.15 (ii). Write $M = l_5$ and let F be the reduced divisor associated to the fiber of type I_4^*. Then the relative log canonical model of the pair $(X, M + aF)$*

1. *contracts every irreducible fiber component not meeting M for $a = 0$*
2. *is an intermediate model for all $0 < a < 1$*
3. *is a twisted model for $a = 1$*

Proof. We use the same notations as in Example I.4.2.15 (ii) and compute:

$$
\begin{aligned}
(K_X + \Delta) \cdot l_{1,4} &= -a \\
(K_X + \Delta) \cdot l_{1,5} &= 1 - a \\
(K_X + \Delta) \cdot l_{3,5} &= 1 - a \\
(K_X + \Delta) \cdot l_{3,6} &= -a \\
(K_X + \Delta) \cdot l_{1} &= a \\
(K_X + \Delta) \cdot l_{1,2} &= 0 \\
(K_X + \Delta) \cdot l_{2} &= 0 \\
(K_X + \Delta) \cdot l_{2,3} &= 0 \\
(K_X + \Delta) \cdot l_{3} &= a
\end{aligned}
$$

where $\Delta \doteq M + aF$.

The computations imply the log canonical model contracts every irreducible fiber component not meeting M for $a = 0$.

If $a > 0$, then there exists a morphism $\mu : X \to X'$ contracting the curves $l_{1,4}$ and

$l_{3,6}$ and we find that

$$\begin{aligned}
(K_{X'} + \Delta') \cdot l'_{1,5} &= 1 - a \\
(K_{X'} + \Delta') \cdot l'_{3,5} &= 1 - a \\
(K_{X'} + \Delta') \cdot l'_{1} &= a/2 \\
(K_{X'} + \Delta') \cdot l'_{1,2} &= 0 \\
(K_{X'} + \Delta') \cdot l'_{2} &= 0 \\
(K_{X'} + \Delta') \cdot l'_{2,3} &= 0 \\
(K_{X'} + \Delta') \cdot l'_{3} &= a/2
\end{aligned}$$

where we have written $D' \doteq \mu_* D$ for any divisor $D \subset X$. In particular, we conclude that the log canonical model is an intermediate model for all $0 < a < 1$ and a twisted model for $a = 1$. Moreover, with the notations introduced in Definition I.4.2.1, $A^{lc} = \varphi^{lc}_*(l_{1,5} + l_{3,5})$ and $E^{lc} = \varphi^{lc}_*(l_1 + l_3)$. □

Lemma I.4.2.19. *Let $X \to \mathbb{P}^1$ be the elliptic K3 surface constructed in Example I.4.2.16 (i). Write $M = l_6$ and let F be the reduced divisor associated to the fiber of type II^*. Then the relative log canonical model of the pair $(X, M + aF)$*

1. *contracts every irreducible fiber component not meeting M for $a = 0$*
2. *is an intermediate model for all $0 < a < 1$*
3. *is a twisted model for $a = 1$*

Proof. Using the same notations as in Example I.4.2.16 (i) we find that

$$
\begin{aligned}
(K_X + \Delta) \cdot l_4 &= -a \\
(K_X + \Delta) \cdot l_{1,5} &= -a \\
(K_X + \Delta) \cdot l_{3,6} &= 1 - a \\
(K_X + \Delta) \cdot l_3 &= a \\
(K_X + \Delta) \cdot l_{3,4} &= 0 \\
(K_X + \Delta) \cdot l_{2,3} &= 0 \\
(K_X + \Delta) \cdot l_2 &= 0 \\
(K_X + \Delta) \cdot l_{1,2} &= 0 \\
(K_X + \Delta) \cdot l_1 &= 0
\end{aligned}
$$

where $\Delta \doteq M + aF$. In particular, we conclude that for $a = 0$ the log canonical model contracts every irreducible fiber component not meeting M. If $a > 0$, we conclude there exists a morphism $\mu : X \to X_1$ contracting the curves l_4 and $l_{1,5}$. By the projection formula, it follows that

$$\begin{aligned}
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_{3,4} &= -a/2 \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_{1} &= -a/2 \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_{3,6} &= 1 - a \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_{3} &= a \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_{2,3} &= 0 \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_{2} &= 0 \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_{1,2} &= 0
\end{aligned}$$

and we see that there exists a morphism $X_1 \to X_2$ further contracting the curves $l_{3,4}$ and l_1. By computing the relevant intersection numbers (as above) one finds that that there exists a sequence of morphisms $X \to X_1 \to X_2 \to \ldots \to X_5$ so that all irreducible curves supported on F get contracted, except $l_{3,6}$ and l_3. Denoting by ψ the composite morphism we compute

$$\begin{aligned}
(K_{X_5} + \psi_*\Delta) \cdot \psi_* l_3 &= a - a/3 - a/6 = a/2 > 0 \\
(K_{X_5} + \psi_*\Delta) \cdot \psi_* l_{3,6} &= 1 - a
\end{aligned}$$

which implies the relative lc model is an intermediate model for all $0 < a < 1$ and it is a twisted model for $a = 1$. Moreover, using the notations introduced in Definition I.4.2.1, it follows that $A^{lc} = \varphi^{lc}_* l_{3,6}$ and $E^{lc} = \varphi^{lc}_* l_3$. □

Lemma I.4.2.20. *Let $X \to \mathbb{P}^1$ be the elliptic K3 surface constructed in Example*

I.4.2.16 (ii). Write $M = l_5$ and let F be the reduced divisor associated to the fiber of type III^*. Then the relative log canonical model of the pair $(X, M + aF)$

1. *contracts every irreducible fiber component not meeting M for $a = 0$*

2. *is an intermediate model for all $0 < a < 1$*

3. *is a twisted model for $a = 1$*

Proof. Using the same notations as in Example I.4.2.16 (ii) we compute:

$$
\begin{aligned}
(K_X + \Delta) \cdot l_{3,4} &= -a \\
(K_X + \Delta) \cdot l_3 &= 0 \\
(K_X + \Delta) \cdot l_{1,3} &= 0 \\
(K_X + \Delta) \cdot l_{1,2} &= 0 \\
(K_X + \Delta) \cdot l_2 &= 0 \\
(K_X + \Delta) \cdot l_1 &= a \\
(K_X + \Delta) \cdot l_{2,5} &= 1 - a \\
(K_X + \Delta) \cdot l_{1,5} &= 1 - a
\end{aligned}
$$

where $\Delta \doteq M + aF$. In particular, for $a = 0$ we see that the log canonical model contracts every irreducible fiber component not meeting M. If $a > 0$ we conclude that there exists a morphism $\mu : X \to X_1$ contracting the curve $l_{3,4}$. Using the projection formula we find that

$$
\begin{aligned}
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_3 &= -a/2 \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_{1,3} &= 0 \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_{1,2} &= 0 \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_2 &= 0 \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_1 &= a \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_{2,5} &= 1-a \\
(K_{X_1} + \mu_*\Delta) \cdot \mu_* l_{1,5} &= 1-a
\end{aligned}
$$

and we see that there exists a morphism $X_1 \to X_2$ further contracting the curve l_3. By proceeding as above and computing the relevant intersection numbers we conclude that there exists a third morphism $X_2 \to X_3$ contracting the curve $l_{1,3}$. If $\psi : X \to X_3$ denotes the composition of such morphisms, then

$$
\begin{aligned}
(K_{X_3} + \psi_*\Delta) \cdot \psi_* l_{1,2} &= 0 \\
(K_{X_3} + \psi_*\Delta) \cdot \psi_* l_2 &= 0 \\
(K_{X_3} + \psi_*\Delta) \cdot \psi_* l_1 &= a - a/4 = 3a/4 > 0 \\
(K_{X_3} + \psi_*\Delta) \cdot \psi_* l_{2,5} &= 1-a \\
(K_{X_3} + \psi_*\Delta) \cdot \psi_* l_{1,5} &= 1-a
\end{aligned}
$$

which implies the log canonical model is an intermediate model for all $0 < a < 1$ and a twisted model for $a = 1$. Moreover, with the notations introduced in Definition I.4.2.1, we have that $A^{lc} = \varphi^{lc}_*(l_{1,5} + l_{2,5})$ and $E^{lc} = \varphi^{lc}_* l_1$. $\square$

Chapter I.5

Classification of relative log canonical models of elliptic surfaces of index two

We now give a complete classification of relative log canonical models of elliptic surface pairs $(f : X \to C, a_M M + aF)$ of index $d_X = 2$, where M is a multisection of degree equals d_X (which exist by Lemma I.2.1.6) and $a_M = 1/d_X = 1/2$.

As in Section I.4.2, we assume F is reduced and unramified, further, we assume $M \cap F$ is supported in the smooth locus of F. Given any such singular fiber we will call one of its components an **end component** of valence n if it corresponds to an end node of valence n in the dual graph of the total geometric fiber[1]. The terminology is needed for Definition I.5.0.5.

[1]For fibers of type II, III and IV we consider a log resolution.

Throughout this chapter we will still call a pair $(f : X \to C, a_M M + aF)$ an elliptic surface pair. Our main result is the following:

Theorem I.5.0.1. *Let $(X, 1/2M + aF)$ be an elliptic surface pair. For any type of fiber other than type I_n there are numbers a_0 and b_0 such that the relative log canonical model*

(i) contracts every irreducible fiber component not meeting M for all $0 \leq a \leq a_0$

(ii) is an intermediate model (see Definition I.4.2.1) for all $a_0 < a < b_0$

(iii) is a twisted model (see Definition I.4.2.1) for all $b_0 \leq a \leq 1$

Moreover, $a_0 = 0$ for fibers of type I_n^*, II^*, III^* and IV^* and $a_0 = lct(X, M, F) = lct(X, F)$ otherwise. Further, if M is special (see Definition I.5.0.5), then $b_0 = a_0 + 1/2(1 - a_0)$. If M is very special (see Definition I.5.0.5), then $b_0 = 0$. Otherwise, $b_0 = 1$.

Remark I.5.0.2. *Theorem I.5.0.1 (i) above for fibers of type II, III or IV has already been proved in Proposition I.4.2.2.*

Remark I.5.0.3. *If F is of type I_n, then we can simply refer to Proposition I.4.2.2 and observe that we have $lct(X, M, F) = lct(X, F) = 1$.*

Remark I.5.0.4. *It is also important to mention that Theorem I.5.0.1 can be easily generalized to the case where the weighted fiber aF is replaced by a weighted sum $\sum a_i F_i$ of marked fibers.*

Such result illustrates the fact that the relative log canonical model of an elliptic surface pair $(X, a_M M + aF)$ depends not only on the type of the fiber F, but also on the geometry of the intersection $M \cap F$ that is, on how the multisection intersects the marked fiber.

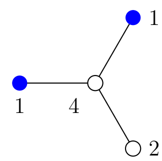
Note that when $b_0 = 1$ our classification agrees with the classification in [2] for the case where the existence of a section is assumed. In fact we will see that the exactly same computations and arguments also apply in some cases, namely the cases where the multisection is assumed to be simple (see Definition I.5.1.1).

We also observe that Theorem I.5.0.1 above is a generalization of Proposition 3.7 in [3].

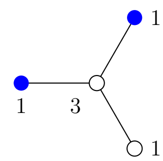
Definition I.5.0.5. *Given an elliptic surface pair $(X, a_M M + aF)$ we say the multisection M is* ***special*** *if $M \cap F$ is supported in two distinct end components of F of valence 1 or $M \cap F$ is supported in a single end component of F of valence two. We say M is* ***very special*** *if M intersects a fiber of type I_n^* only at components of multiplicity 2.*

Remark I.5.0.6. *Note that the definition above depends both on M and on F. For instance, it excludes elliptic pairs with marked fiber of type I_n or II.*

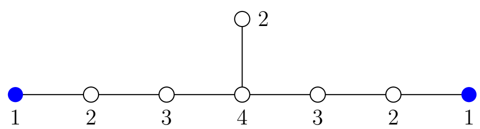
We illustrate in the diagrams below all the possible components (colored) of all the different types of fibers F that can meet a special multisection (of degree 2).

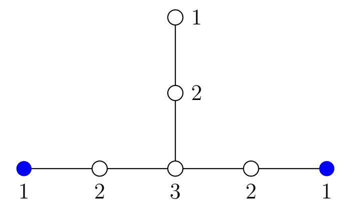


Type III



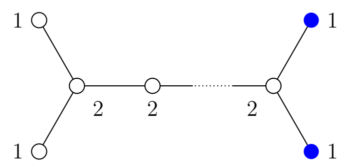
Type IV

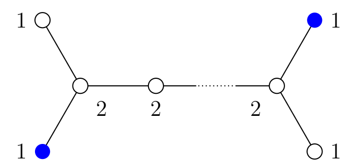




Type III^*

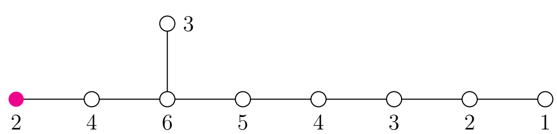
Type IV^*

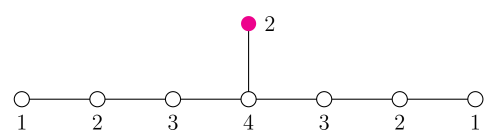




Type I_n^*

Type I_n^* and $n \geq 1$





Type II^*

Type III^*

Remark I.5.0.7. *Note that for a fiber of type III the component of multiplicity 2 is part of the exceptional divisor in the log resolution and therefore it cannot intersect the multisection. We are assuming $M \cap F$ is supported in the smooth locus of F.*

In order to prove Theorem I.5.0.1 we prove a series of lemmas. Each of these lemmas assumes a particular configuration for the intersection $M \cap F$ and by proving those lemmas we cover all possible configurations.

The strategy in our proofs consists in simply running the log MMP and it is summarized next. We start by computing the intersection numbers

$$(K_X + a_M M + aF) \cdot \gamma$$

for any γ an irreducible curve supported on a fiber. Note that because we are only interested in describing the boundary divisor in the relative log canonical model, it suffices to consider simply those curves which are supported on F.

If all these numbers are non-negative, then $K_X + \Delta$ is $f-$nef, where $\Delta \doteq a_M M + aF$. By abundance, it follows that $K_X + \Delta$ is $f-$semiample and the relative log canonical model contracts precisely those curves γ such that $(K_X + \Delta) \cdot \gamma = 0$. If for some γ the number $(K_X + \Delta) \cdot \gamma$ is negative, then there exists a morphism $\mu : X \to X'$ contracting γ. We then repeat the first step applied to the pair (X', Δ'), where $\Delta' \doteq \mu_* \Delta$ and we proceed this way until there are no curves γ for which the numbers $(K_X + \Delta) \cdot \gamma$ are negative.

Given an elliptic surface pair $(X, a_M M + aF)$ we fix the notation and will denote by A the divisor supported in the components of F meeting the multisection M.

I.5.1 The relative lc model when M is simple

Definition I.5.1.1. *Given an elliptic surface pair $(X, a_M M + aF)$ we say the multisection M is* ***simple*** *if A is irreducible and reduced that is, A is an end component of valence 1.*

Lemma I.5.1.2 (M simple)**.** *Let $(X, a_M M + aF)$ be an elliptic surface pair, where $a_M = 1/2$ and we assume M is simple (of degree 2). Then we can find a number a_0 so that the relative log canonical model*

(i) contracts every irreducible fiber component not meeting M for all $0 \leq a \leq a_0$

(ii) is an intermediate model for all $a_0 < a < 1$

(iii) is a twisted model for $a = 1$

Moreover, $a_0 = 0$ for fibers of type I_n^*, II^*, III^* and IV^* and $a_0 = lct(X, M, F) = lct(X, F)$ otherwise.

Remark I.5.1.3. *In the statement above we don't necessarily need to assume $m = 2$, the proof below works for any $m \in \mathbb{Z}_{>1}$.*

Proof. If F is of type II, III or IV, then the pair $(X, a_M M + aF)$ is not log canonical for all a and we need to first take a log resolution $\varphi : (Z, \tilde{M} + a\tilde{F} + \mathrm{Exc}(\varphi)) \to (X, M + aF)$. As before, we write $\tilde{F}$ (resp. $\tilde{M}$) to denote the strict transform (under φ) of F (resp. M) and we mark the exceptional divisor $\mathrm{Exc}(\varphi)$ with coefficient one.

We then compute the intersection numbers $(K_Z + \tilde{\Delta}) \cdot \tilde{\gamma}$ for $\tilde{\gamma}$ an irreducible component of $\varphi^* F$ and $\tilde{\Delta} \doteq a_M \tilde{M} + a\tilde{F} + \mathrm{Exc}(\varphi)$. We observe that the exactly same computations and arguments from [2] apply here, hence their results.

In fact the same is true for the other types of fiber. For fibers of type I_n^*, II^*, III^* or IV^* the pair $(X, a_M M + aF)$ is already log canonical and by computing $(K_X + \Delta) \cdot \gamma$ for γ one of the irreducible components of F and $\Delta \doteq a_M M + aF$ we see that again the arguments in [2] apply. This happens precisely because we are marking the multisection M with a coefficient $a_M = 1/m = 1/\deg M$ and we are assuming M is simple. $\square$

I.5.2 The relative lc model when M is special

Lemma I.5.2.1 (M special). *Let $(X, 1/2M + aF)$ be an elliptic surface pair and assume M is special. Then we can find numbers a_0 and b_0 such that the relative log canonical model*

(i) contracts every irreducible fiber component not meeting M for all $0 \leq a \leq a_0$

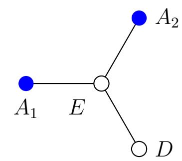
(ii) is an intermediate model for all $a_0 < a < b_0$

(iii) is a twisted model for all $b_0 \leq a \leq 1$

Moreover, $a_0 = 0$ for fibers of type I_n^*, II^*, III^* and IV^* and $a_0 = lct(X, M, F) = lct(X, F)$ otherwise. Further, $b_0 = a_0 + 1/2(1 - a_0)$.

Proof. First we observe that our definition of a special multisection excludes elliptic pairs with marked fiber of type I_n or II. Next, we consider F of type III or IV. For such types of fiber we need to first take a log resolution $\varphi : (Z, \tilde{M} + a\tilde{F} + \mathrm{Exc}(\varphi)) \to (X, M + aF)$. As before, we write $\tilde{F}$ (resp. $\tilde{M}$) to denote the strict transform (under φ) of F (resp. M) and we mark the exceptional divisor $\mathrm{Exc}(\varphi)$ with coefficient one.

For such types of fibers $\varphi^* F$ has dual graph



where the blue nodes mark the component $A = A_1 + A_2$ that meets the multisection M.

In the table below we summarize the multiplicities and self intersections of the various components for each type of fiber. We also indicate the components of $\tilde{F}$ and the components of $\mathrm{Exc}(\varphi)$.

Type	$\tilde{F}$	$\mathrm{Exc}(\varphi)$	Mult(D)	Mult(E)	A_i^2	D^2	E^2
III	$A_1 + A_2$	$D + E$	2	4	-4	-2	-1
IV	$A_1 + A_2 + D$	E	1	3	-3	-3	-1

Now, if F is of type III, then $K_Z = \varphi^* K_X + D + 2E$ so that

$$
\begin{aligned}
(K_Z + \tilde{\Delta}) \cdot A_i &= \frac{7 - 8a}{2} \\
(K_Z + \tilde{\Delta}) \cdot D &= -1 \\
(K_Z + \tilde{\Delta}) \cdot E &= 2a - 1
\end{aligned}
$$

where $\tilde{\Delta} \doteq 1/2\tilde{M} + a\tilde{F} + \mathrm{Exc}(\varphi)$. The above computation implies that there exists a morphism $\mu : Z \to Z'$ contracting D and E whenever $0 \leq a \leq 1/2$ and the relative log canonical model is just the pair $(X, 1/2M + aF)$ itself. If $1/2 < a \leq 3/4$, there exists a morphism $\mu : Z \to Z'$ contracting D and we find, by the projection formula, that

$$
\begin{aligned}
(K'_Z + \Delta') \cdot A'_i &= \frac{7-8a}{2} \\
(K'_Z + \Delta') \cdot E' &= 2a - 3/2
\end{aligned}
$$

where $\Delta' \doteq \mu_* \tilde{\Delta}$ and we write $A'_i \doteq \mu_* A_i$ and so on. Again we can further contract E' and the relative log canonical model is just the pair $(X, 1/2M + aF)$ itself. Moreover, note that the latter computation tells us that for $3/4 < a < 7/8$ the relative lc model is an intermediate model with fiber $A' + E'$, where $A' \doteq A'_1 + A'_2$. It also gives us that for $7/8 \leq a \leq 1$ we have a twisted model that is, we can further contract A'.

Note that $a_0 \doteq 3/4$ and $b_0 \doteq 7/8$ are related by $b_0 = a_0 + 1/2(1 - a_0)$. Moreover, $a_0 = lct(X, M, F) = lct(X, F)$.

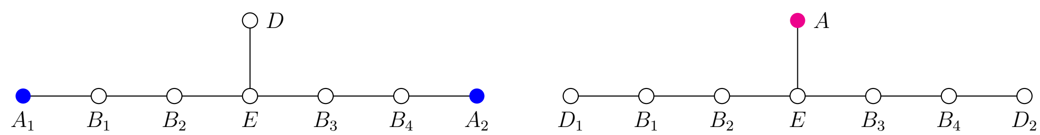
The computations for a type IV fiber are similar and we omit the details. In that case we have $K_Z = \varphi^* K_X + E$ and the relevant intersection numbers we need to compute as the first step when running the MMP are given below:

$$
\begin{aligned}
(K_Z + \tilde{\Delta}) \cdot A_i &= \frac{5-6a}{2} \\
(K_Z + \tilde{\Delta}) \cdot D &= 2 - 3a \\
(K_Z + \tilde{\Delta}) \cdot E &= 3a - 2
\end{aligned}
$$

Note that again we have that $a_0 \doteq 2/3 = lct(X, M, F) = lct(X, F)$ and $b_0 \doteq 5/6$ are related by $b_0 = a_0 + 1/2(1 - a_0)$.

We also omit the computations for a fiber of type II^*. Such computations are almost the same as those presented in the proof of Lemma **??**. We simply need to replace M by $1/2M$.

Next, let us assume F is of type III^*. Then there are two possible configurations for $M \cap F \subset A$ and to fix some notation we label the various components of the dual graph of F in each configuration as indicated below



As before the colored nodes mark the component A that meets the multisection M. Note that in both cases the pair $(X, 1/2M + aF)$ is already log canonical since X is smooth and the divisor $\Delta \doteq 1/2M + aF$ is normal crossings.

The relevant intersection numbers are computed below. In the first case (left) we have

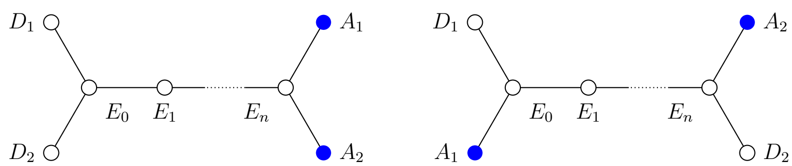
$$\begin{aligned}
(K_X + \Delta) \cdot A_i &= 1/2 - a \\
(K_X + \Delta) \cdot B_j &= 0 \\
(K_X + \Delta) \cdot D &= -a \\
(K_X + \Delta) \cdot E &= a
\end{aligned}$$

whereas in the second case (right) we find:

$$\begin{aligned}
(K_X + \Delta) \cdot A &= 1/2 - a \\
(K_X + \Delta) \cdot B_j &= 0 \\
(K_X + \Delta) \cdot D_i &= -a \\
(K_X + \Delta) \cdot E &= a
\end{aligned}$$

Nonetheless we see that the numbers $a_0 \doteq 0$ and $b_0 \doteq 1/2$ once more satisfy the equation $a_0 = b_0 - 1/2(1 - a_0)$.

Similarly, if F is of type I_n^* there are also two cases to be considered. The corresponding dual graphs for F are illustrated below



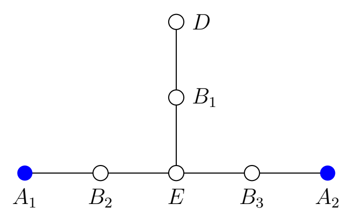
The notation we need is also indicated by the labellings in the diagrams. In both cases we compute

$$\begin{aligned}
(K_X + \Delta) \cdot A_i &= 1/2 - a \\
(K_X + \Delta) \cdot D_i &= -a \\
(K_X + \Delta) \cdot E_j &= a \qquad \text{for } j = 0, n \\
(K_X + \Delta) \cdot E_k &= 0 \qquad \text{for } k = 1, \ldots, n-1
\end{aligned}$$

where $\Delta \doteq 1/2M + aF$. The main difference lies in the fact that in the first case (left) the reduced component E^{lc} of the fiber of both an intermediate model and a twisted model is irreducible and is given by the image of E_n under $\varphi^{lc} : X \to X^{lc}$. On the other hand, in the second case (right), the corresponding component is no longer irreducible and is given by the image of $E_0 + E_n$ (under φ^{lc}). This is a new phenomena, which doesn't appear in the classification of [2] for surfaces of index one (with a section).

For a fiber of type I_n^* and M special the numbers a_0 and b_0 are 0 and 1/2, respectively. Again they satisfy the equation $a_0 = b_0 - 1/2(1 - a_0)$.

Finally, consider F a fiber of type IV^*. The support of such fiber consists of seven (-2) rational curves and has dual graph an affine E_6. We label each component as indicated below



The blue nodes mark the component $A = A_1 + A_2$ meeting the multisection M.

The relevant intersection numbers in this case are:

$$
\begin{aligned}
(K_X + \Delta) \cdot A_i &= 1/2 - a \\
(K_X + \Delta) \cdot B_j &= 0 \\
(K_X + \Delta) \cdot D &= -a \\
(K_X + \Delta) \cdot E &= a
\end{aligned}
$$

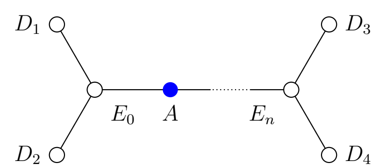
where, as before, we write $\Delta \doteq 1/2M + aF$. Note that, once more, the numbers

$a_0 = 0$ and $b_0 = 1/2$ verify $a_0 = b_0 - 1/2(1 - a_0)$. $\square$

I.5.3 The relative lc model when M is very special

Lemma I.5.3.1 (F of type I_n^* and M very special)**.** *Let $(X, 1/2M + aF)$ be an elliptic surface pair with F of type I_n^* and assume M is very special. Then the relative log canonical model is a twisted model for all $0 \leq a \leq 1$.*

Proof. If F is of type I_n^*, then it consists of $n+5$ components which are (-2) rational curves arranged in a way so that the dual graph is an affine D_{n+4}. If M is very special, then it intersects one of the multiplicity two components $E_0, \ldots, E_n$ as illustrated below.



That is, the component A meeting the multisection M agrees with one of the E_i, for some $0 \leq i \leq n$.

The relevant intersection numbers are as follows:

$$
\begin{aligned}
(K_X + \Delta) \cdot A &= 1/2 + a \qquad \text{if } i = 0 \text{ or } i = n \\
(K_X + \Delta) \cdot A &= 1/2 \qquad \text{if } i \neq 0, n \\
(K_X + \Delta) \cdot D_j &= -a \\
(K_X + \Delta) \cdot E_k &= 0 \qquad \text{if } k \neq i \text{ and } 1 \leq k \leq n-1 \\
(K_X + \Delta) \cdot E_l &= a \qquad \text{if } l = 0 \text{ or } l = n \text{ and } l \neq i
\end{aligned}
$$

where $\Delta \doteq 1/2M + aF$.

Now, if $a = 0$, then the relative lc model contracts all components except A. And in fact we have that $a_0 = b_0 = 0$: If $a > 0$, then there exists a morphism $\mu : X \to X'$ contracting all the components labeled by D_j. Writing $D' \doteq \mu_* D$ for any divisor $D \subset X$ we find that

$$
\begin{aligned}
(K'_X + \Delta') \cdot A' &= 1/2 \\
(K'_X + \Delta') \cdot E'_k &= 0 \\
(K'_X + \Delta') \cdot E'_l &= 0
\end{aligned}
$$

That is, $K'_X + \Delta'$ is $f'-$nef, hence semiample (by abundance) and the log canonical model further contracts all components different than $A' = E'_i$. In other words, the (relative) lc model is a twisted model for all $0 \leq a \leq 1$. $\square$

I.5.4 The relative lc model when M is exotic

Definition I.5.4.1. *Given an elliptic surface pair* $(X, a_M M + aF)$ *we say the multisection* M *is* ***exotic*** *if* M *is neither special nor very special nor simple.*

Lemma I.5.4.2 (M exotic)**.** *Let* $(X, 1/2M + aF)$ *be an elliptic surface pair with* F *not of type* I_n *and assume* M *is exotic. Then we can find a number* a_0 *so that the relative log canonical model*

(i) contracts every irreducible fiber component not meeting M *for all* $0 \leq a \leq a_0$

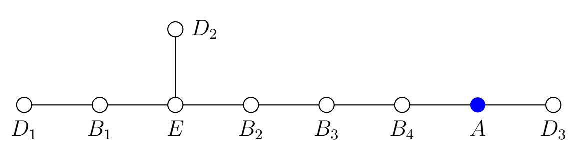
(ii) is an intermediate model for all $a_0 < a < 1$

(iii) is a twisted model for $a = 1$

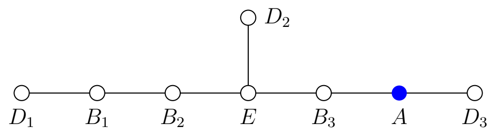
Moreover, $a_0 = 0$ *for fibers of type* II^*, III^* *and* IV^* *and* $a_0 = 1$ *for a fiber of type* I_n*.*

Proof. First, note that if an elliptic pair $(X, 1/2M + aF)$ is such that M is different, then, by definition, F is not of type II, III, IV or I_n^*.

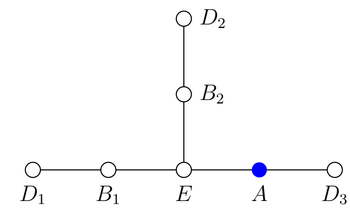
For all the other possible types of fiber F we illustrate in the diagrams below their dual graphs. The components meeting the multisection are marked by the blue nodes and some extra notation is also introduced.



Type II^*



Type III^*



Type IV^*

In any case the relevant intersection numbers are (where $\Delta \doteq 1/2M + aF$)

$$
\begin{aligned}
(K_X + \Delta) \cdot A &= 1/2 \\
(K_X + \Delta) \cdot B_i &= 0 \\
(K_X + \Delta) \cdot D_j &= -a \\
(K_X + \Delta) \cdot E &= a
\end{aligned}
$$

If $a = 0$, then the log canonical model contracts all the components except the component A that meets the multisection. If $a > 0$, then there exists a morphism $\mu : X \to X'$ contracting the components labeled by D_j. Writing $D' \doteq \mu_* D$ for any divisor $D \subset X$ we find that $(K'_X + \Delta') \cdot A' = \dfrac{1-a}{2}$. Moreover, $(K'_X + \Delta') \cdot E' = a/2$ if F is of type II^* or III^* and $(K'_X + \Delta') \cdot E' = a$ if F is of type IV^*.

In particular, the conclusion is that the relative log canonical model is (i) an intermediate model for $0 < a < 1$ that is, we can further contract all the components labeled by B'_i; or (ii) it is a twisted model for $a = 1$ that is, we also contract A'.

$\square$

Chapter I.6

Classification of relative lc models of elliptic threefolds

Let $f_0 : X_0 \to S_0$ be an elliptic fibration with section from an irreducible threefold X_0 to a surface S_0 and such that the generic fiber is a smooth elliptic curve. Then one can construct a smooth model $f : X \to S$ for the elliptic threefold X_0 as in [41] satisfying:

(i) X and S are smooth;

(ii) f is flat and minimal;

(iii) the discriminant locus $D \subset S$ is normal crossings;

(iv) at a smooth point $p \in D$, the singular fiber $f^{-1}(p)$ is of Kodaira type, with the fiber type being locally constant near p; and

(v) at a singular point $p \in D$, the singular fiber $f^{-1}(p)$ is birationally determined (see also Remark I.6.0.1 below) by the two types of singular fibers over the two branches of D at p.

Remark I.6.0.1. *It is conjectured in* [21] *(Conjecture 9.8) that Kodaira's classification from Table I.2.1 actually extends to the class of* ***birationally equivalent*** *relatively minimal elliptic threefolds. And, moreover, the classification can be obtained by associating to the discriminant locus the non abelian gauge algebras and their representations as in* [21, Section 8].

We will call such model, $f : X \to S$, a "Miranda smooth model". Following [41], we assume that $S_0 = S$ so that (locally) at a double point p of the reduced discriminant locus we can write a minimal Weierstrass equation (for X_0):

$$y^2 = x^3 + s_1^{L_1} s_2^{L_2} ax + s_1^{K_1} s_2^{K_2} b \tag{I.6.0.1}$$

where a and b are local units at p and $L_i \leq 3$ or $K_i \leq 5$ for $i = 1, 2$.

In particular, over the two branches $R_i : (s_i = 0)$ the generic fibers have type (L_i, K_i, N_i). Throughout this section we will write $F_i \doteq f^{-1}(R_i)_{\text{red}}$ and we will use Miranda's terminology and say that we have a collision $F_1 + F_2$. Our choice of taking the divisors F_i reduced will be justified in Proposition I.6.1.1.

Given a section $\sigma : S \to X$ we will write S instead of $\sigma(S)$ and given weights $0 \leq a, b \leq 1$, we will refer to the pair $(X, S + aF_1 + bF_2)$ as an **elliptic threefold pair**. Similarly, we will write $(W, S' + aF_1' + bF_2')$ for the corresponding pair in the (minimal) Weierstrass model. That is, $F_i' \doteq \varphi_* F_i$ and $S' \doteq \varphi_* S$, where $\varphi : X \to W$ is the birational map defining W.

For all the possible collisions that Miranda considers in [41] φ is actually a minimal crepant resolution of the singularities of W.

The goal of this chapter is to give a classification of relative log canonical models of elliptic threefold pairs (with respect to the fibration morphism). In particular, it makes sense to ask when is a pair $(X, S + aF_1 + bF_2)$ as above a log canonical pair.

Following Miranda analysis in [41] we observe that $F_1 \cap F_2$ is normal crossings, which tells us a log resolution of the pair $(X, S + aF_1 + bF_2)$ is given by taking first a log resolution of $(X, S + aF_1)$ followed by a log resolution of $(X, S + bF_2)$. It is also important to note that by assumption $S \cap F_i$ is a smooth point. As a consequence, a straightforward computation of the relevant log discrepancies gives us:

Proposition I.6.0.2. *Let $(X, S + aF_1 + bF_2)$ be an elliptic threefold pair. Such pair is log canonical if and only if $0 \leq a \leq lct(X, F_1)$ and $0 \leq b \leq lct(X, F_2)$*

Proof. Let $\pi : Z \to X$ be a log resolution given by taking first a log resolution of $(X, S + aF_1)$ followed by a log resolution of $(X, S + bF_2)$. Denote by $\tilde{F}_i$ the strict transform of F_i. We know that we can write

$$K_Z = \pi^* K_X + \sum a_k E_k$$

for some a_k and some divisors E_k. Similarly, there are some coefficients b_i and c_j so that

$$\pi^* F_1 = \tilde{F}_1 + \sum b_i E_i \qquad \text{and} \qquad \pi^* F_2 = \tilde{F}_2 + \sum c_j E_j$$

This data allows us to compute the log canonical threshold for the pairs (X, F_i) and therefore also for the pair $(X, S + aF_1 + bF_2)$. Note that it is enough to consider the case $a, b > 0$. In particular we have

$$lct(X, F_1) = \min\left\{\frac{1+a_i}{b_i}, 1\right\} \qquad \text{and} \qquad lct(X, F_2) = \min\left\{\frac{1+a_j}{c_j}, 1\right\}$$

while

$$lct(X, S + aF_1 + bF_2) = \min\left\{\frac{1+a_i}{ab_i}, \frac{1+a_j}{bc_j}, 1\right\}$$

since by our choice of a log resolution we have $E_i \neq E_j$ for all i, j.

As a consequence, the pair $(X, S + aF_1 + bF_2)$ is log canonical if and only if $1 \leq \min\left\{\frac{1+a_i}{ab_i}, \frac{1+a_j}{bc_j}\right\}$ if and only if $a \leq \min\left\{\frac{1+a_i}{b_i}\right\}$ and $b \leq \min\left\{\frac{1+a_j}{c_j}\right\}$.

Note that the log canonical threshold is a birational invariant and does not depend on the choice of a log resolution. □

Remark I.6.0.3. *Note that using the same notation as in the proof of the previous Proposition we find that*

$$lct(X, F_1 + F_2) = \min\left\{\frac{1+a_i}{b_i}, \frac{1+a_j}{c_j}, 1\right\} = \min\{lct(X, F_1), lct(X, F_2)\}$$

since $E_i \neq E_j$ for all i, j.

Remark I.6.0.4. *Note also that by the previous Remark if $0 \leq a, b \leq lct(X, F_1 + F_2)$, then the pair $(X, S + aF_1 + bF_2)$ is log canonical.*

Remark I.6.0.5. *Corollary I.3.0.9 gives an alternative proof for the "forward direction" by taking $Y = X$, $\Delta' = S$ and $\Delta = aF_1 + bF_2$.*

Corollary I.6.0.6. *Let* $(X, S+aF_1+bF_2)$ *be an elliptic threefold pair, where* F_1+F_2 *is a divisor of fibers given by a collision* I_n+I_m*, with* n *and* m *not both odd, or* $II+IV$*. If* $0 \leq a \leq lct(X, F_1)$ *and* $0 \leq b \leq lct(X, F_2)$*, then the pair* $(W, S' + aF_1' + bF_2')$ *is log canonical.*

Proof. This follows from the fact that for such type of collisions we have $lct(X, F_i) = lct(W, F_i')$ and hence

$$lct(X, S + aF_1 + bF_2) = lct(W, S' + aF_1' + bF_2')$$

□

Proposition I.6.0.7. *Let* $(X, S+aF_1+bF_2)$ *be an elliptic threefold pair, where* F_1+F_2 *is a divisor of fibers given by a collision* $I_n + I_m$*, with* n *and* m *not both odd. Then the pair* $(W, S' + aF_1' + bF_2')$ *is the relative log canonical model for all* $0 \leq a, b \leq 1$ *(See also Section I.6.1).*

Proof. Note that the pair $(W, S' + aF_1' + bF_2')$ satisfies Definition I.3.1.1 since we can apply Lemma I.3.0.10 to $\varphi : X \to W$ and, moreover, for any irreducible curve γ supported on a fiber of $f' : W \to S$ we have

$$(K_W + S' + aF_1' + bF_2') \cdot \gamma = 1$$

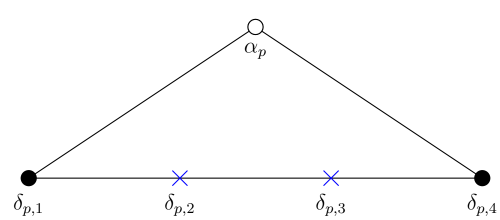
That is, the divisor $K_W + S' + aF_1' + bF_2'$ is $f'-$ample. □

It is important to observe that the Miranda smooth model is constructed by first resolving the singularities over the R_1 component. In particular, we get different configurations for the fiber over the double point depending on which of the numbers n and m one calls n and which m. Although we have this non-unicity of the central fiber, the result above tells us the relative log canonical model for collisions $I_n + I_m$ does not depend on the central fiber. A fact that will be latter verified for any type of collision in Theorem I.6.0.15.

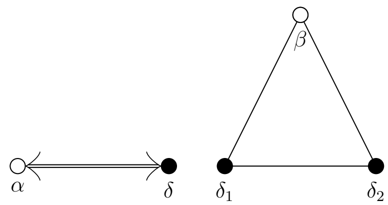
The example below illustrates this non-unicity phenomenon. We consider the collision $I_3 + I_2$, meaning over R_1 (resp. R_2) we have fibers of type I_3 (resp. I_2), and also the collision $I_2 + I_3$, where now I_2 and I_3 are interchanged. Similarly, the latter is the same as choosing to first resolve the singularities over R_2 (for the collision $I_3 + I_2$).

It is interesting to observe that in such example the two possible models are related by "Atiyah flops". In fact in both models the exceptional locus over the double point is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (marked in the diagram below by $\times$). Now, if we fix one of these models, then each of the two rational curves can be flopped, yielding the other model.

Example I.6.0.8 ($I_3 + I_2$ vs. $I_2 + I_3$)**.** *Consider an elliptic threefold pair $(X, S + aF_1 + bF_2)$, where $F_1 + F_2$ is given by a collision $I_3 + I_2$ or $I_2 + I_3$. We obtain two different configurations for the central fiber of Miranda's smooth model, each having five components:*



One configuration is given by choosing I_3 (resp. I_2) as the type of the generic fiber *over R_1 (resp. R_2). The other one is then given by interchanging I_2 and I_3. The diagrams below represent the dual graphs for such types of fiber and establish some notation we will use:*



For the first possible configuration we find that

$$
\begin{aligned}
\alpha_p &\sim \alpha \\
1/2\delta_{p,1} + 1/2\delta_{p,4} &\sim \delta \\
\delta_{p,2} + \delta_{p,3} + \alpha_p &\sim \beta \\
1/2\delta_{p,1} &\sim \delta_1 \\
1/2\delta_{p,4} &\sim \delta_2
\end{aligned}
$$

whereas for the second configuration we have

$$
\begin{aligned}
\alpha_p &\sim \beta \\
\delta_{p,1} + \delta_{p,2} &\sim \delta_1 \\
\delta_{p,3} + \delta_{p,4} &\sim \delta_2 \\
1/2\delta_{p,1} + 1/2\delta_{p,4} + \alpha_p &\sim \alpha \\
\delta_{p,2} + \delta_{p,3} &\sim \delta
\end{aligned}
$$

In both cases the exceptional locus for $X \dashrightarrow X_0$ consists of the pair of rational *curves $\delta_{p,2} + \delta_{p,3}$ (in blue). In the first configuration such curves have $\mathcal{O} \oplus \mathcal{O}(-2)$ normal bundles whilst in the second one, they have $\mathcal{O}(-1)^{\oplus 2}$.*

We now observe that the exactly same argument as in the proof of Proposition I.6.0.9 also applies to the following:

Proposition I.6.0.9. *Let $(X, S{+}aF_1{+}bF_2)$ be an elliptic threefold pair, where $F_1{+}F_2$ is a divisor of fibers given by a collision $II + IV$. Then the pair $(W, S' + aF_1' + bF_2')$ is the relative log canonical model for all $0 \leq a \leq 5/6$ and $0 \leq b \leq 2/3$ (See also Section I.6.1).*

A natural question to ask then is: What happens for $5/6 < a \leq 1$ or $5/6 < b \leq 1$? Theorem I.6.0.15 gives us a complete description of the relative log canonical model as the weights a and b vary.

As we have already mentioned, in the computation of the relative log canonical model, i.e., in running the log MMP, we do not need to know what the central fiber over p is. Moreover, given an elliptic threefold pair $(X, S + aF_1 + bF_2)$ we can find numbers a_0, b_0 so that the relative log canonical model is

(i) the Weierstrass model for $0 \leq a \leq a_0$ and $0 \leq b \leq b_0$,

(ii) the intermediate model (see Definition I.6.0.18) for $a_0 < a < 1$ and $b_0 < b < 1$,

(iii) the twisted model (see Definition I.6.0.18) for $a = 1$ and $b = 1$.

The numbers a_0 and b_0, as one could expect from Proposition I.6.0.2, are birational invariants and are given by $a_0 \doteq lct(X, F_1)$ and $b_0 \doteq lct(X, F_2)$.

Of course these are not all the possibilities for the weights a and b, but we see that we have found a somewhat analogous description to the surface case as in [2].

Note that the above description implies that if we take $a = b$, then the relative log canonical model of the pair $\big(X, S + a(F_1 + F_2)\big)$ is

(i) the Weierstrass model for $0 \leq a \leq c_0 \doteq \min\{a_0, b_0\} = lct(X, F_1 + F_2)$,

(ii) the intermediate model for $\max\{a_0, b_0\} < a < 1$,

(iii) the twisted model for $a = 1$.

The next Proposition considers the particular case of a collision $I_n + I_m^*$ and a more careful analysis of its proof is the key ingredient for proving the corresponding general statement for any type of collision (Theorem I.6.0.15).

Proposition I.6.0.10. *Given $(X, S + aF_1 + bF_2)$ an elliptic threefold pair, where $F_1 + F_2$ is a divisor of fibers given by a collision $I_n + I_m^*$, let $\varphi^{lc} : (X, S + aF_1 + bF_2) \to (Y, S^{lc} + F_{a,b}^{lc})$ be the relative log canonical model, where we write $\Delta^{lc} \doteq \varphi_*^{lc}\Delta$ for Δ a divisor on X. Then $(Y, S^{lc} + F_{a,b}^{lc})$ is*

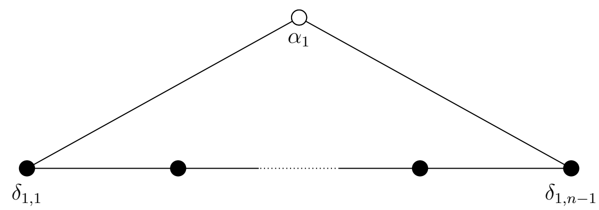
(i) the (minimal) Weierstrass model for any $0 \leq a \leq 1$ and $b = 0$,

(ii) given by $F_{a,b}^{lc} = aA_1^{lc} + b(A_2^{lc} + E_2^{lc})$ for any $0 \leq a \leq 1$ and $0 < b < 1$. That is, if $0 \leq a \leq 1$ and $0 < b < 1$, then φ^{lc} contracts all divisors over R_1 except A_1 and it contracts all divisors over R_2 except A_2 and $E_{2,0}$. Every fiber component not supported on the F_i and not meeting the section is also contracted.

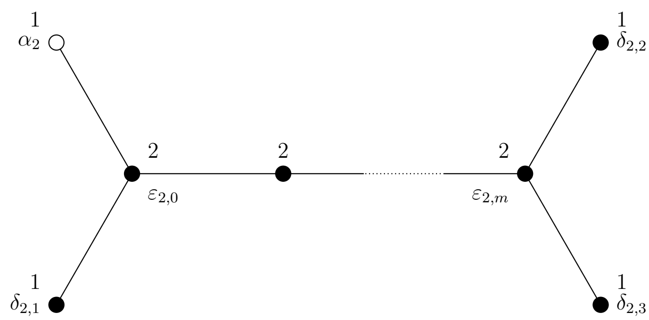
(iii) *given by* $F^{lc}_{a,b} = aA^{lc}_1 + E^{lc}_2$ *for* $0 \leq a \leq 1$ *and* $b = 1$*. That is, if* $0 \leq a \leq 1$ *and* $b = 1$*, then* φ^{lc} *contracts all divisors over* R_1 *except* A_1 *and it contracts all divisors over* R_2 *except* $E_{2,0}$*. Every fiber component not supported on the* F_i *and not meeting the section is also contracted.*

Proof. First, we establish the notation for the irreducible components of the generic and special fibers over the R_i as indicated in the diagrams below representing its duals graphs

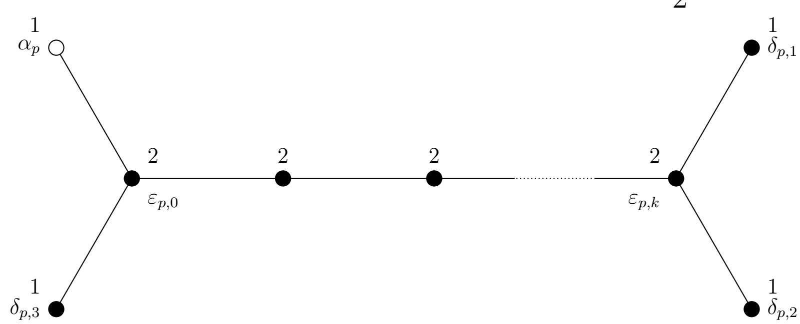
Generic fiber over R_1



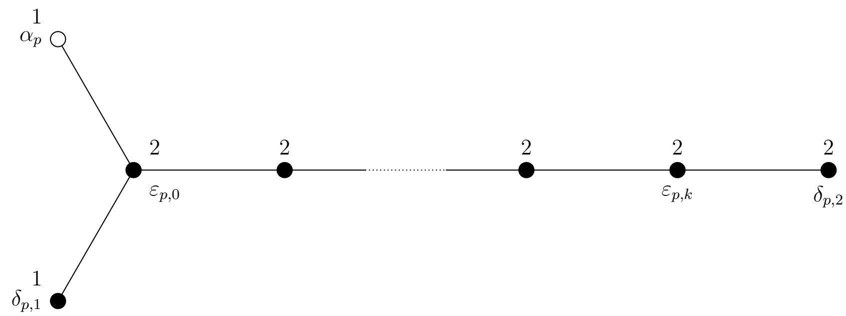
Generic fiber over R_2



Fiber over p if n is even, where $k = m + \dfrac{n}{2}$.



Fiber over p if n is odd, where $k = m + \frac{n-1}{2}$.



Note that over the I_n component there are n' divisors $A_1, D_{1,1}, \ldots, D_{1,n'-1}$, where $n' = \frac{n}{2} + 1$ if n is even and $n' = \frac{n+3}{2}$ if n is odd. Over the I_m^* component there are $m+5$ divisors $A_1, D_{2,1}, D_{2,2}, D_{2,3}, E_{2,0}, \ldots, E_{2,m}$ if m is even and $m+4$ divisors $A_1, D_{2,1}, D_{2,2}, E_{2,0}, \ldots, E_{2,m}$ if m is odd. The central fibers of each of these divisors and a more detailed description can be found in the Appendix of [22].

The next step is then to compute the intersection numbers

$$(K_X + S + aF_1 + bF_2) \cdot \gamma$$

for any irreducible curve γ supported in a fiber of $f : X \to S$. We find that there exists a birational map $\mu : X \to \bar{X}$ contracting all the curves $\delta_{2,i}$ (and $\delta_{p,i}$), hence the divisors $D_{2,i}$ over R_2.

Writing $\bar{\Delta} \doteq \mu_* \Delta$ for any divisor Δ on X and $\bar{\gamma} \doteq \mu_* \gamma$ for any curve γ on X we conclude that if $0 < b < 1$, the morphism $\varphi^{lc} : \bar{X} \to X^{lc}$ leaves only the divisors $A_i^{lc} \doteq \varphi_*^{lc} \bar{A}_i$ and $E_2^{lc} \doteq \varphi_*^{lc} \bar{E}_{2,0}$. When $b = 1$ the morphism φ^{lc} also contracts $\bar{A}_2$.

Finally, if $b = 0$, the morphism φ^{lc} contracts all components except the A_i and the relative canonical model is the Weierstrass model.

Remark I.6.0.11. *Note that if γ is not supported in the F_i, then $\gamma \cdot F_i = 0$, so that*

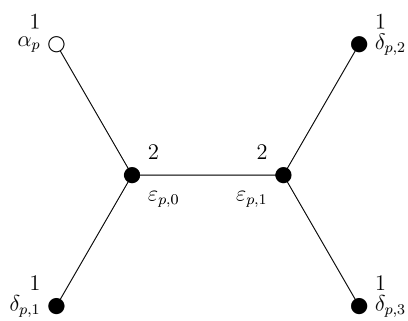
$$(K_X + S + aF_1 + bF_2) \cdot \gamma = 1 \qquad \text{if } \gamma \text{ meets the section}$$

and $(K_X + S + aF_1 + bF_2) \cdot \gamma = 0$ otherwise. In particular, in the log canonical model all such curves not meeting the section are contracted as well.

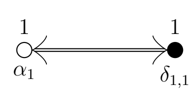
□

The example below illustrates the kind of computations that were omitted in the proof of Proposition I.6.0.10.

Example I.6.0.12. *[Explicit computation] Let us consider the case of an $I_2 + I_0^*$ collision. The central fiber of Miranda's smooth model has six components:*



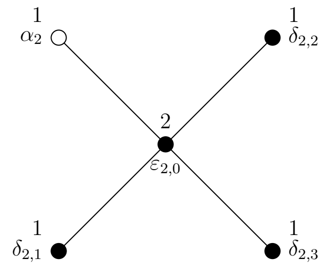
Over R_1, that is the I_2 component, there are two divisors A_1 and $D_{1,1}$ whose central fibers are $\alpha_p + \delta_{p,1} + 2\varepsilon_{p,0}$ and $\delta_{p,2} + \delta_{p,3} + 2\varepsilon_{p,1}$, respectively. In particular, if the generic fiber over R_1 is given by



it follows that

$$\begin{aligned} \alpha_1 &\sim \alpha_p + \delta_{p,1} + 2\varepsilon_{p,0} \\ \delta_{1,1} &\sim \delta_{p,2} + \delta_{p,3} + 2\varepsilon_{p,1} \end{aligned}$$

On the other hand, over R_2, that is, in the I_0^* component, there are five divisors $A_2, D_{2,1}, D_{2,2}, D_{2,3}$ and $E_{2,0}$ whose central fibers are α_p, $\delta_{p,1}$, $\delta_{p,2}$, δ_{p3} and $\varepsilon_{p,0} + \varepsilon_{p,1}$, respectively. As a consequence, if the generic fiber over R_2 is given by



we have that

$$
\begin{aligned}
\alpha_2 &\sim \alpha_p \\
\delta_{2,i} &\sim \delta_{p,i} \qquad \text{for } i = 1, 2, 3 \\
\varepsilon_{2,0} &\sim \varepsilon_{p,0} + \varepsilon_{p,1}
\end{aligned}
$$

The above data allows us to compute the intersection numbers

$$(K_X + S + aF_1 + bF_2) \cdot \gamma$$

for any irreducible curve γ supported in a fiber over R_1 and/or R_2:

$$
\begin{aligned}
(K_X + S + aF_1 + bF_2) \cdot \alpha_1 &= 1 \\
(K_X + S + aF_1 + bF_2) \cdot \delta_{1,1} &= 0 \\
(K_X + S + aF_1 + bF_2) \cdot \alpha_2 &= 1 - b \\
(K_X + S + aF_1 + bF_2) \cdot \delta_{2,i} &= -b \\
(K_X + S + aF_1 + bF_2) \cdot \varepsilon_{2,0} &= 2b \\
(K_X + S + aF_1 + bF_2) \cdot \alpha_p &= 1 - b \\
(K_X + S + aF_1 + bF_2) \cdot \delta_{p,i} &= -b \\
(K_X + S + aF_1 + bF_2) \cdot \varepsilon_{p,1} &= b \\
(K_X + S + aF_1 + bF_2) \cdot \varepsilon_{p,0} &= b
\end{aligned}
$$

Now, from the computations above we see that there exists a birational map $\mu : X \to \bar{X}$ contracting all the curves $\delta_{p,i}$ and $\delta_{2,i}$, hence the divisors $D_{2,i}$ over R_2.

Writing $\bar{\Delta} \doteq \mu_* \Delta$ for any divisor Δ on X and $\bar{\gamma} \doteq \mu_* \gamma$ for any curve γ on X we compute:

$$
\begin{aligned}
(K_{\bar{X}} + \bar{S} + a\bar{F}_1 + b\bar{F}_2) \cdot \bar{\varepsilon}_{p,0} = \frac{1}{2} b \\
(K_{\bar{X}} + \bar{S} + a\bar{F}_1 + b\bar{F}_2) \cdot \bar{\varepsilon}_{p,1} = 0 \\
(K_{\bar{X}} + \bar{S} + a\bar{F}_1 + b\bar{F}_2) \cdot \bar{\varepsilon}_{2,0} = \frac{1}{2} b
\end{aligned}
$$

The conclusion is that of Proposition I.6.0.10.

Remark I.6.0.13. *It is interesting to observe that the computations in the previous example (and more generally for $I_n + I_0^*$) do not depend on whether the curves $\delta_{2,i}$ are independent or not as homology classes.*

Remark I.6.0.14. *Note also that whether or not φ^{lc} contracts a curve in the fiber over p is completely determined by some combination of irreducible curves supported in the generic fibers of the components of the F_i.*

The previous remark is the key on understanding why the relative log canonical of an elliptic threefold pair $(X, S + aF_1 + bF_2)$ does not depend on the central fiber.

After possibly taking a log resolution $\pi : (Z, \tilde{S} + a\tilde{F}_1 + b\tilde{F}_2 + \mathrm{Exc}(\pi)) \to (X, S + aF_1 + bF_2)$ we get a log canonical pair. Now, each irreducible component over each of the branches R_i has the structure of a $\mathbb{P}^1$-bundle and, therefore, any two fibers are numerically equivalent. In particular, writing $\pi^{-1}(F_i)_{\mathrm{red}}$ as a union of irreducible divisors:

$$\pi^{-1}(F_1)_{\mathrm{red}} = Y_{1,1} \cup \ldots \cup Y_{1,k_1} \qquad \pi^{-1}(F_2)_{\mathrm{red}} = Y_{2,1} \cup \ldots \cup Y_{2,k_2}$$

we conclude that when running the log MMP we only need to consider the generic fibers of the divisors $Y_{i,j}$ $(i = 1, 2)$. That is, we only need to consider the positivity of

$$(K_Z + \tilde{S} + a\tilde{F}_1 + b\tilde{F}_2 + \mathrm{Exc}(\pi)) \cdot \gamma_{i,j}$$

for all $\gamma_{i,j}$ generic fibers of the divisors $Y_{i,j}$. The log canonical model contracts a divisor $Y_{i,j}$ if and only if it contracts its generic fiber.

A nice consequence is that the computations become completely analogous as the ones found in [2] for the surface case.

Since we already have a complete description of the relative log canonical model for the collisions $I_n + I_m$ and $I_n + I_m^*$, by means of such analogy, we write A_i for the unique component of $\tilde{F}_i$ meeting the section $\tilde{S}$ and we denote by E_i the component of $\pi^{-1}(F_i)$ with the highest multiplicity. For all other possible collisions these are well defined.

With such notations, given $(X, S + aF_1 + bF_2)$ an elliptic threefold pair, let

$$\varphi^{lc} : (X, S + aF_1 + bF_2) \to (Z, \tilde{S} + a\tilde{F}_1 + b\tilde{F}_2 + \mathrm{Exc}(\pi)) \underset{\psi}{\to} (Y, S^{lc} + F^{lc}_{a,b})$$

denote the relative log canonical model. Writing $\Delta^{lc} \doteq \psi_* \Delta$ for any divisor Δ on Z we can describe the divisor $F^{lc}_{a,b}$ completely:

Theorem I.6.0.15. *Given $(X, S + aF_1 + bF_2)$ an elliptic threefold pair, there are numbers a_0, b_0 and c_0 so that the relative log canonical model $(Y, S^{lc} + F^{lc}_{a,b})$ is given by:*

(i) the (minimal) Weierstrass model for any $0 \leq a \leq a_0$ and $0 \leq b \leq b_0$

In particular, for any $0 \leq a, b \leq c_0 \doteq \min\{a_0, b_0\}$. (See also Section I.6.1)

(ii) $F^{lc}_{a,b} = aA^{lc}_1 + E^{lc}_1 + b(A^{lc}_2 + E^{lc}_2)$ for $a_0 < a < 1$ and $b_0 < b < 1$

(iii) $F^{lc}_{a,b} = aA^{lc}_1 + b(A^{lc}_2 + E^{lc}_2)$ for $0 \leq a \leq a_0$ and $b_0 < b < 1$

(iv) $F^{lc}_{a,b} = aA^{lc}_1 + bE^{lc}_2$ for $0 \leq a \leq a_0$ and $b = 1$

(v) $F^{lc}_{a,b} = aA^{lc}_1 + E^{lc}_1 + bA^{lc}_2$ *for* $a_0 < a < 1$ *and* $0 \leq b \leq b_0$

(vi) $F^{lc}_{a,b} = aA^{lc}_1 + E^{lc}_1 + bE^{lc}_2$ *for* $a_0 < a < 1$ *and* $b = 1$

(vii) $F^{lc}_{a,b} = E^{lc}_1 + bA^{lc}_2$ *for* $a = 1$ *and* $0 \leq b \leq b_0$

(viii) $F^{lc}_{a,b} = E^{lc}_1 + b(A^{lc}_2 + E^{lc}_2)$ *for* $a = 1$ *and* $b_0 < b < 1$

(ix) $F^{lc}_{a,b} = E^{lc}_1 + E^{lc}_2$ *for* $a = 1$ *and* $b = 1$

for any type of collision, except the collision $II + IV$*. For the collision* $II + IV$ *the divisor* E^{lc}_2 *appears with coefficient one in the above description. Moreover, the numbers* a_0, b_0 *and* c_0 *are given by the following table*

Collision	a_0	b_0	c_0
$I_n + I_m$	1	1	1
$I_n + I^*_m$	1	0	0
$II + IV$	5/6	2/3	2/3
$II + I^*_0$	5/6	0	0
$II + IV^*$	5/6	0	0
$IV + I^*_0$	2/3	0	0
$III + I^*_0$	3/4	0	0

Remark I.6.0.16. *For the collision* $I_n + I_m$ *there are no divisors* E_i *and for* $I_n + I^*_m$ *there is no* E_1*, but we take* $E_2 = E_{2,0}$ *as in the proof of Proposition I.6.0.10.*

Remark I.6.0.17. *The divisor E_2^{lc} appears with coefficient one for the collision $II+IV$ because F_2 is not normal crossings and we actually have $E_2 \subset Exc(\pi)$.*

For the models described by (ii) and (ix) in Theorem I.6.0.15 we use the same terminology as introduced in [2]:

Definition I.6.0.18. *Given a log canonical model $(Y, S^{lc}+F_{a,b}^{lc})$ of an elliptic threefold pair, we call it a* ***twisted model*** *if $F_{a,b}^{lc}$ is irreducible but non-reduced (case (ix)). We call it an* ***intermediate model*** *if $F_{a,b}^{lc}$ is a normal crossings union of a reduced divisor $A \doteq A_1^{lc}+A_2^{lc}$ and a non-reduced component $E = E_1^{lc}+E_2^{lc}$ such that the section meets the fibers along the smooth locus of A (case (ii)). We note that in the twisted model the section meets the fibers along singular points of the total space.*

Remark I.6.0.19. *Note that, at least locally, we can view $(X, S+aF_1+bF_2)$ as a family of elliptic surface pairs over $Spec(\mathbb{C}[t]) \simeq \mathbb{A}^1$. For instance, we can identify S with $Spec(\mathbb{C}[s_1, s_2])$ and consider it as a family of marked curves $s_2 = s_1+t$, with markings at $s_1 = 0$ and $s_2 = 0$. Then, what Theorem I.6.0.15 says is that the relative lc model of the pair $(X, S+aF_1+bF_2)$, viewed as a family, has as its fibers the relative lc models of the fibers of the family $(X \to S \to \mathbb{A}^1, S+aF_1+bF_2)$.*

As a consequence of Theorem I.6.0.15 we obtain an analogue of [3, Theorem 3.10]. We can classify relative log canonical models of (minimal) Weierstrass threefold pairs $(W, S'+aF_1'+bF_2')$, where $W = X_0$ is a (minimal) Weierstrass threefold given by an equation as in (I.6.0.1), $S' = S$ is a section and $F_i' = f_0^{-1}(R_i)$ (see notations in the beginning of Section I.6).

That is, given a pair $(W, S' + aF_1' + bF_2')$ and a log resolution $p : Z \to W$ we can classify the log canonical model of $(Z, p_*^{-1}(S' + aF_1' + bF_2') + Exc(p))$ relative to $f' \circ p : Z \to S$, where we take $W = X_0$ and $f' = f_0$.

For the collisions $II + IV$ and $I_n + I_m$ this agrees with the log canonical model of the pair $(X, aF_1 + bF_2)$ relative to $f : X \to S$, which is given by Theorem I.6.0.15 because for those types of collision we are already marking the divisor $E \doteq E_1^{lc} + E_2^{lc}$ with coefficient one. In fact for the collision $II + IV$ we can take (Z, p) so that it fits into a commutative diagram:

$$\begin{tikzcd}
& (Z, p_*^{-1}(S' + aF_1' + bF_2') + \mathrm{Exc}(p)) \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\
(X, S + aF_1 + bF_2) \arrow[rr, "\varphi"] & & (W, S' + aF_1' + bF_2')
\end{tikzcd}$$

In general we find:

Theorem I.6.0.20. *Given $(W, S' + aF_1' + bF_2')$ as above let $(Y, S^{lc} + F_{a,b}^{lc})$ be its relative log canonical model. For any type of collisions there are numbers a_0, b_0 and c_0 so that the relative log canonical model is*

(i) the (minimal) Weierstrass model for any $0 \leq a \leq a_0$ and $0 \leq b \leq b_0$

In particular, for any $0 \leq a, b \leq c_0 \doteq \min\{a_0, b_0\}$. (See also Section I.6.1)

(ii) $F_{a,b}^{lc} = aA_1^{lc} + E_1^{lc} + bA_2^{lc} + E_2^{lc}$ for $a_0 < a < 1$ and $b_0 < b < 1$

(iii) $F_{a,b}^{lc} = aA_1^{lc} + bA_2^{lc} + E_2^{lc}$ for $0 \leq a \leq a_0$ and $b_0 < b < 1$

(iv) $F_{a,b}^{lc} = aA_1^{lc} + E_2^{lc}$ for $0 \leq a \leq a_0$ and $b = 1$

(v) $F^{lc}_{a,b} = aA^{lc}_1 + E^{lc}_1 + bA^{lc}_2$ *for* $a_0 < a < 1$ *and* $0 \leq b \leq b_0$

(vi) $F^{lc}_{a,b} = aA^{lc}_1 + E^{lc}_1 + E^{lc}_2$ *for* $a_0 < a < 1$ *and* $b = 1$

(vii) $F^{lc}_{a,b} = E^{lc}_1 + bA^{lc}_2$ *for* $a = 1$ *and* $0 \leq b \leq b_0$

(viii) $F^{lc}_{a,b} = E^{lc}_1 + bA^{lc}_2 + E^{lc}_2$ *for* $a = 1$ *and* $b_0 < b < 1$

(ix) $F^{lc}_{a,b} = E^{lc}_1 + E^{lc}_2$ *for* $a = 1$ *and* $b = 1$

Moreover, the numbers a_0, b_0 *and* c_0 *are birational invariants and are given by* $a_0 = lct(W, F'_1), b_0 = lct(W, F'_2)$ *and* $c_0 = \min\{a_0, b_0\} = lct(W, F'_1 + F'_2)$ *as indicated in the table below:*

Collision	a_0	b_0	c_0
$I_n + I_m$	1	1	1
$I_n + I^*_m$	1	1/2	1/2
$II + IV$	5/6	2/3	2/3
$II + I^*_0$	5/6	1/2	1/2
$II + IV^*$	5/6	1/3	1/3
$IV + I^*_0$	2/3	1/2	1/2
$III + I^*_0$	3/4	1/2	1/2

The notation in Theorem I.6.0.20 is as follows.

We write A_i for the unique component of $p^{-1}_*(F'_i)$ meeting the section and we denote by E_i the unique exceptional divisor of p that intersects A_i except for the collision $II + IV^*$, where E_2 is the component of Exc(p) with the highest multiplicity and E_1 is still defined as before.

Remark I.6.0.21. *Note that, as in Theorem I.6.0.15, for the collision $I_n + I_m$ there are no divisors E_i and for $I_n + I_m^*$ there is no E_1.*

With such notations, given $(W, S' + aF_1' + bF_2')$, if

$$\varphi^{lc} : (W, S' + aF_1' + bF_2') \to (Z, p_*^{-1}(S' + aF_1' + bF_2') + Exc(p)) \underset{\psi'}{\to} (Y, S^{lc} + F_{a,b}^{lc})$$

denotes the relative log canonical model, Theorem I.6.0.20 describes the divisor $F_{a,b}^{lc}$ completely: We write $\Delta^{lc} \doteq \psi'_*\Delta$ for any divisor Δ on Z.

I.6.1 The non-reduced case

We now consider the case where the marked divisor $F_1 + F_2$ is possibly non-reduced. That is, with the same notations from the previous paragraphs, we take $F_i \doteq f^{-1}(R_i)$ instead of considering its associated reduced divisor. We find that:

Proposition I.6.1.1. *Given an elliptic threefold pair $(f : X \to S, S + aF_1 + bF_2)$ (as in the previous section), with weights $0 \le a \le a_0 \doteq lct(X, F_1)$ and $0 \le b \le b_0 \doteq lct(X, F_2)$, the relative log canonical model is the minimal Weierstrass model independent of the type of collision.*

Proof. Note that the choice of the weights is such that $(X, S + aF_1 + bF_2)$ is a log pair (in fact log canonical). Note also that $lct(X, F_i) = lct(W, F_i')$. Now, we know that $\varphi : X \to W$ is a minimal crepant resolution, hence

$$K_X + S + aF_1 + bF_2 = \varphi^*(K_W + S' + aF_1' + bF_2')$$

where $S' + aF_1' + bF_2' = \varphi_*(S + aF_1 + bF_2)$. In particular, by Lemma I.3.0.10, we have

$$a(E, X, S + aF_1 + bF_2) = a(E, W, S' + aF_1' + bF_2')$$

for any $\varphi-$exceptional divisor. But then, the pair $(W, S' + aF_1' + bF_2')$ satisfies Definition I.3.1.1, since

$$(K_W + S' + aF_1' + bF_2') \cdot \gamma = 1 > 0$$

for any irreducible curve γ supported on a fiber of $f' : W \to S$. That is, $K_W + S' + aF_1' + bF_2'$ is $f'-$ample. □

Remark I.6.1.2. *Note that by taking* $0 \leq a, b \leq c_0 \doteq \min\{a_0, b_0\} = lct(X, F_1 + F_2)$ *we have that* $0 \leq a \leq a_0$ *and* $0 \leq b \leq b_0$.

Remark I.6.1.3. *Note that at least for* $0 \leq a \leq a_0$ *and* $0 \leq b \leq b_0$ *we can recover our previous results for the collisions* $I_n + I_m$ *and* $II + IV$ *since in those cases the marked divisors are already reduced.*

I.6.2 Non-Miranda type collisions

The possible collisions considered by Miranda in [41] are such that the corresponding smooth models $f : X \to S$ are actually flat. Moreover, the birational morphism $\varphi : X \dashrightarrow W$ from the smooth model X to the (minimal) Weierstrass model W is a crepant resolution of the singularities of W.

In this section we consider collisions $II + II$ and $IV + IV$. These can still be described by the corresponding equations as in (I.6.0.1), but when resolving the singularities of $X_0 = W$ the resulting smooth model $f : X \to S$ no longer satisfies the above mentioned properties. More precisely, f is no longer flat (although its generic fiber is still an elliptic curve) and $\varphi : X \dashrightarrow W$ is no longer crepant. In fact, for these two types of collisions we have that $K_X = \varphi^* K_W + E$ and, further, $E \subset f^{-1}(p)$, where p (as before) denotes the double point in the discriminant locus.

In particular, we find:

Proposition I.6.2.1. *Let $(X, S+aF_1+bF_2)$ be an elliptic threefold pair, where F_1+F_2 is a divisor of fibers given by a collision $II+II$, or $IV+IV$. Then the relative log canonical model is the (minimal) Weierstrass model for any $0 \leq a, b \leq lct(X, F_1+F_2)$. Moreover, $lct(X, F_1+F_2) = lct(X, F_i)$*

Proof. Let $\Delta \doteq S + aF_1 + bF_2$ and consider $\varphi : X \to W$ the birational map from X to the (minimal) Weierstrass model. Define $\Delta' \doteq \varphi_* \Delta$. Then X and W fit into a commutative diagram as in Lemma I.3.0.11 with $X' = Y = W$ and $\varphi' = id_W$. In particular, the pair (W, Δ') satisfies Definition I.3.1.1 whenever $0 \leq a, b \leq lct(X, F_1 + F_2)$. $\square$

Now, an analogue of Proposition I.6.0.2 still holds for a pair $(X, S + aF_1 + bF_2)$ as in Proposition I.6.2.1. In particular, if $lct(X, F_1 + F_2) < a \leq 1$, then the pair

$(X, S + aF_1 + bF_2)$ is not log canonical and we need to take a log resolution π : $(Z, \tilde{S} + a\tilde{F_1} + b\tilde{F_2} + \mathrm{Exc}(\pi)) \to (X, S + aF_1 + bF_2)$. Since $f^{-1}(p)$ is normal crossings, we can obtain Z by first taking a log resolution of (X, F_1), followed by a log resolution of (X, F_2). Note that the section S still meets F_i transversally and at smooth points.

Since $(K_Z + \tilde{S} + a\tilde{F_1} + b\tilde{F_2} + \mathrm{Exc}(\pi)) \cdot \gamma < 0$ for any curve γ supported on $\pi^{-1}(E)$, it follows that a divisor in Z is contracted by the log canonical model if and only if its generic fiber gets contracted. Again, we conclude that the relative canonical model of a pair $(X, S + aF_1 + bF_2)$ as above does not depend on the central fiber.

We have obtained the following:

Theorem I.6.2.2. *Let $(X, S + aF_1 + bF_2)$ be an elliptic threefold pair, where $F_1 + F_2$ is a divisor of fibers given by a collision $II + II$ or $IV + IV$. Then the relative log canonical model is*

(i) the (minimal) Weierstrass model for any $0 \leq a, b \leq lct(X, F_1 + F_2)$

(ii) the "intermediate model" for any $lct(X, F_1 + F_2) < a, b < 1$

(iii) the "twisted model" for $a = b = 1$

For all other possibilities of values of a and b the conclusion is the same as for Theorem I.6.0.15.

Part II

Stability of pencils of plane curves

Chapter II.1

Introduction

In this second part of the thesis we study the problem of classifying pencils of curves of degree d in $\mathbb{P}^2$ using geometric invariant theory. The results presented here consist of the content of three papers [54–56], which we reorganize in two chapters.

In Chapter II.2 we consider the action of $SL(3)$ and we relate the stability of a pencil of plane curves to the stability of its generators, to the log canonical threshold of its members, and to the multiplicities of its base points, thus obtaining explicit stability criteria.

Letting $\mathscr{P}_d$ denote the space of all pencils of plane curves of degree d, our main results are given by Theorems II.1.0.1, II.1.0.2 and II.1.0.3 below.

Theorem II.1.0.1 ([55]). *Let $\mathcal{P}$ be a pencil in $\mathscr{P}_d$ containing a curve C_f such that $lct(\mathbb{P}^2, C_f) = \alpha$. If $\mathcal{P}$ is unstable (resp. not stable), then $\mathcal{P}$ contains a curve C_g such that $lct(\mathbb{P}^2, C_g) < \dfrac{3\alpha}{2d\alpha - 3}$ (resp. $\leq$).*

Theorem II.1.0.2 ([55]). *If $\mathcal{P} \in \mathscr{P}_d$ is semistable (resp. stable), then $lct_p(\mathbb{P}^2, C_f) \geq \frac{3}{2d}$ (resp. $>$) for any curve C_f in $\mathcal{P}$ and any base point p.*

Theorem II.1.0.3 ([55]). *Let $\mathcal{P}$ be a pencil in $\mathscr{P}_d$. If we can find two generators C_f and C_g of $\mathcal{P}$ such that $mult_p(C_f) + mult_p(C_g) > \frac{4d}{3}$ (resp. $\geq$) for some base point p, then $\mathcal{P}$ is unstable (resp. not stable).*

One of the ingredients in our approach consists in observing that we can sometimes determine whether a pencil $\mathcal{P} \in \mathscr{P}_d$ is (semi)stable or not by looking at the stability of its generators. We also prove Theorems II.1.0.4, II.1.0.5 and II.3.2.15 below:

Theorem II.1.0.4 ([55]). *If a pencil $\mathcal{P} \in \mathscr{P}_d$ has only semistable (resp. stable) members, then $\mathcal{P}$ is semistable (resp. stable).*

Theorem II.1.0.5 ([55]). *If $\mathcal{P} \in \mathscr{P}_d$ contains at worst one strictly semistable curve (and all other curves in $\mathcal{P}$ are stable), then $\mathcal{P}$ is stable.*

Theorem II.1.0.6 ([55]). *If $\mathcal{P} \in \mathscr{P}_d$ contains at worst two semistable curves C_f and C_g (and all other curves in $\mathcal{P}$ are stable), then $\mathcal{P}$ is strictly semistable if and only if there exists a one-parameter subgroup λ (and coordinates in $\mathbb{P}^2$) such that C_f and C_g are both non-stable with respect to this λ.*

In Chapter II.3, we then use these criteria and the results obtained in [54] to provide a complete and geometric characterization of the stability of certain pencils of plane sextics called Halphen pencils of index two (Definition II.3.1.4). Inspired by [40], we provide a description of their stability in terms of the type of singular fibers appearing in the associated rational elliptic surfaces (see Proposition II.3.1.9).

The results obtained by Miranda in [40] say that the stability of a pencil $\mathcal{P}$ of plane cubics is completely determined by the type of singular fibers F occurring in the corresponding rational elliptic surface (with section). Here we prove the following two theorems, which have the same flavor:

Theorem II.1.0.7 ([56])**.** *Let $\mathcal{P}$ be a Halphen pencil of index two, which we write as $\lambda B + \mu(2C) = 0$, where C is the unique cubic through the nine base points and B corresponds to a (non-multiple) fiber F of the associated rational elliptic surface $f : Y \to \mathbb{P}^1$.*

When C is smooth $\mathcal{P}$ is stable if and only if one of the following statements hold:

(i) all fibers of Y are reduced

(ii) Y contains at most one non-reduced fiber F of type I_n^* or IV^*

(iii) there exists exactly one (non-multiple) fiber F in Y of type II^* or III^* and B is semistable

(iv) Y contains two fibers of type I_0^* and there is no one-parameter subgroup λ that destabilizes the two corresponding curves simultaneously.

Similarly, when C is singular $\mathcal{P}$ is stable if and only if one of the following statements hold:

(i') all fibers of Y are reduced

(ii') $\mathcal{P}$ contains at worst two strictly semistable curves and there is no one-parameter subgroup λ that destabilizes these two curves simultaneously

(iii') Y contains a fiber of type IV^* and B is unstable

Theorem II.1.0.8 ([56]). *$\mathcal{P}$ is semistable if and only if either every curve in $\mathcal{P}$ is semistable or Y does not contain a fiber F of type II^*.*

Our approach to prove Theorems II.1.0.7 and II.1.0.8 has three main ingredients:

1) the explicit constructions of Halphen pencils in Appendix A (or [54]) and the classification from Theorems A.1.4, A.1.5 and A.1.6 (= [54, Theorem 1.2])

2) the inequalities provided by Theorem II.1.0.9 below and

3) Theorems II.1.0.1 and II.1.0.2, which relate the stability of a pencil of plane curves of degree d to the log canonical threshold.

Theorem II.1.0.9 ([54]). *If M_B (resp. M_F) denotes the largest multiplicity of a component of B (resp. F), then*

(i) $lct(\mathbb{P}^2, B) \leq \dfrac{1}{M_B} \leq 2lct(Y, F)$

(ii) if F is reduced, then $\dfrac{1}{2} < lct(\mathbb{P}^2, B) \leq lct(Y, F)$

(iii) if $M_F \geq 2$ and F is not reduced, then $lct(Y, F) \leq lct(\mathbb{P}^2, B)$

In particular, we observe that an important ingredient is the study of the singularities of a plane curve occurring in a Halphen pencil – with the log canonical threshold (lct) playing an important role. In fact we establish a dictionary (Section II.3.1.1) between the curves in a Halphen pencil of index two and the fibers in the associated rational elliptic surfaces.

Chapter II.2

Stability of pencils of plane curves, log canonical thresholds and multiplicities

Hacking [23] and Kim-Lee [27] observed the following simple connection between two notions of stability, one coming from geometric invariant theory (GIT) and the other coming from the MMP: They observed that if $H \subset \mathbb{P}^n$ is a hypersurface of degree d and the pair $\left(\mathbb{P}^n, \dfrac{n+1}{d} H\right)$ is log canonical, then H is GIT semistable for the natural action of $PGL(n+1)$. And if $\left(\mathbb{P}^n, (\dfrac{n+1}{d} + \varepsilon) H\right)$ is log canonical for some $0 < \varepsilon \ll 1$, then H is stable.

In this chapter (and in [55]) we relate the GIT stability of a pencil $\mathcal{P}$ of plane curves of degree d under the action of $SL(3)$ to the log canonical threshold of pairs

$(\mathbb{P}^2, \mathcal{C}_d)$, where $\mathcal{C}_d$ is a curve in $\mathcal{P}$. Part of our approach consists in observing that we can partially determine whether a pencil $\mathcal{P} \in \mathscr{P}_d$ is unstable (resp. not stable) or not by looking at the stability of its generators. Moreover, adapting the ideas in [11, Lemma 3.3], we are also able to relate the GIT stability of a pencil $\mathcal{P}$ to the multiplicities of its base points. When $d = 6$ we also consider a different approach (the same as in [40]) and obtain a complete description of the stability criteria (Section II.2.5).

II.2.1 An overview of geometric invariant theory

We first recall the relevant definitions and results from Geometric Invariant Theory, and we point the reader to [15] for more details.

The setup consists of a reductive group G acting on an algebraic variety X and we start by first assuming $X \simeq \mathbb{C}^{n+1}$

Definition II.2.1.1. *A point $x \in X$ is said to be semistable for the $G-$action if an only if $0 \notin \overline{G \cdot x}$.*

Definition II.2.1.2. *A point $x \in X$ is said to be stable for the $G-$action if and only if the following two conditions hold:*

(i) The orbit $G \cdot x \subset X$ is closed and

(ii) The stabilizer $G_x \leq G$ is finite

Definition II.2.1.3. *If $X \hookrightarrow \mathbb{P}^n$ is a projective variety, a point $x \in X$ will be called semistable (resp. stable) if any point $\tilde{x} \in \mathbb{C}^{n+1}$ lying over x is semistable (resp. stable).*

From now on we assume that this is the case.

Definition II.2.1.4. *A one-parameter subgroup of G consists of a non-trivial group homomorphism $\lambda : \mathbb{C}^\times \to G$.*

Given a one-parameter subgroup $\lambda : \mathbb{C}^\times \to G$ we may regard $\mathbb{C}^{n+1}$ as a representation of $\mathbb{C}^\times$. Since any representation of $\mathbb{C}^\times$ is completely reducible and every irreducible representation is one dimensional, we can choose a basis $e_0, \ldots, e_n$ of $\mathbb{C}^{n+1}$ so that $\lambda(t) \cdot e_i = t^{r_i} e_i$, for some $r_i \in \mathbb{Z}$. Then, given $x \in X \hookrightarrow \mathbb{P}^n$ we can pick $\tilde{x} \in Cone(X) \subset \mathbb{C}^{n+1}$ lying above x and write $\tilde{x} = \sum x_i e_i$ with respect to this basis so that $\lambda(t) \cdot x \doteq \lambda(t) \cdot \tilde{x} = \sum t^{r_i} x_i e_i$. The weights of x are the set of integers r_i for which x_i is not zero.

Definition II.2.1.5. *Given $x \in X$ we define the Hilbert-Mumford weight of x at λ to be $\mu(x, \lambda) \doteq \min\{r_i \ : \ x_i \neq 0\}$.*

Remark II.2.1.6. *The Hilbert-Mumford weight satisfies the following properties:*

(i) $\mu(x, \lambda^n) = n\mu(x, \lambda)$ *for all* $n \in \mathbb{N}$

(ii) $\mu(g \cdot x, g\lambda g^{-1}) = \mu(x, \lambda)$ *for all* $g \in G$

The known numerical criterion for stability can thus be stated:

Theorem II.2.1.7 (Hilbert-Mumford criterion)**.** *Let G be a reductive group acting linearly on a projective variety $X \hookrightarrow \mathbb{P}^n$. Then for a point $x \in X$ we have that x is semistable (resp. stable) if and only if $\mu(x, \lambda) \leq 0$ (resp. $<$) for all one-parameter subgroups λ of G.*

That is, a point $x \in X$ is unstable (resp. not stable) for the $G-$action if and only if there exists a one-parameter subgroup $\lambda : \mathbb{C}^\times \to G$ for which all the weights of x are all positive (resp. non-negative).

II.2.2 Stability criterion for pencils of plane curves

As in [40], we view a pencil of plane curves of degree d as a choice of line in the space of all plane curves of degree d. In other words, we identify the space $\mathscr{P}_d$ of all such pencils with the Grassmannian $Gr(2, S^dV^*)$, where $V \doteq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. The latter, in turn, can be embedded in $\mathbb{P}(\Lambda^2 S^d V^*)$ via Plücker coordinates. The group $SL(V)$ acts naturally on V, hence on the invariant subvariety $\mathscr{P}_d$, and our goal is to describe the corresponding GIT stability conditions. Since our main tool for that is criterion of Hilbert-Mumford, we need to know how the diagonal elements act on such coordinates.

Concretely, choosing a pencil $\mathcal{P} \in \mathscr{P}_d$ and two curves C_f and C_g as generators, these represented (in some choice of coordinates) by $f = \sum f_{ij} x^i y^j z^{d-i-j} = 0$ and $g = \sum g_{ij} x^i y^j z^{d-i-j} = 0$ respectively, the Plücker coordinates of $\mathcal{P}$ are given by all the 2×2 minors

$$m_{ijkl} \doteq \begin{vmatrix} f_{ij} & f_{kl} \\ g_{ij} & g_{kl} \end{vmatrix}$$

Thus, the action of $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \in SL(V)$ on the Plücker coordinates is given by

$$(m_{ijkl}) \mapsto (\alpha^{i+k}\beta^{j+l}\gamma^{2d-i-j-k-l}m_{ijkl})$$

So we can now express the Hilbert-Mumford criterion for a pencil $\mathcal{P} \in \mathscr{P}_d$ as the vanishing of some of its Plücker coordinates (m_{ijkl}) with respect to a convenient choice of basis. In view of Remark II.2.1.6 (ii) we assume any one-parameter subgroup λ is normalized, meaning we choose coordinates $[x, y, z]$ in $\mathbb{P}^2$ so that we have

$$\begin{array}{rcl} \lambda : \mathbb{C}^\times & \to & SL(V) \\ t & \mapsto & \left([x,y,z] \mapsto \begin{pmatrix} t^{a_x} & 0 & 0 \\ 0 & t^{a_y} & 0 \\ 0 & 0 & t^{a_z} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) \end{array} \tag{II.2.2.1}$$

for some weights $a_x, a_y, a_z \in \mathbb{Z}$ satisfying $a_x \geq a_y \geq a_z, a_x > 0$ and $a_x + a_y + a_z = 0$.

In particular, the action of $\lambda(t)$ in the Plücker coordinates is given by

$$(m_{ijkl}) \mapsto (t^{e_{ijkl}}m_{ijkl})$$

where $e_{ijkl} = e_{ijkl}(\lambda) \doteq a_x(2i + 2k + j + l - 2d) + a_y(2j + 2l + i + k - 2d)$.

The sign of the function $\mu(\mathcal{P}, \lambda)$ does not change under these reductions and the Hilbert-Mumford criterion becomes:

Proposition II.2.2.1. *A pencil $\mathcal{P} \in \mathscr{P}_d$ is unstable (resp. not stable) if and only if there exists a one-parameter subgroup λ and coordinates in $\mathbb{P}^2$ such that if the pencil is represented in those coordinates by (m_{ijkl}), then $m_{ijkl} = 0$ whenever $e_{ijkl}(\lambda) \leq 0$ (resp. < 0).*

II.2.2.1 The stability of the generators

It turns out that we are able to partially determine whether a pencil $\mathcal{P} \in \mathscr{P}_d$ is unstable (resp. not stable) or not by looking at the stability of its generators and, in particular, by looking at the log canonical threshold of its members. Therefore, from now on we will consider the actions of $SL(V)$ on both $\mathscr{P}_d$ and the space of plane curves of degree d.

Our strategy consists in introducing an "affine" analogue of the Hilbert-Mumford weight (see Definition II.2.5.1) and translate the numerical criterion of Hilbert-Mumford in terms of this quantity. More precisely, given a pencil $\mathcal{P} \in \mathscr{P}_d$ and a curve $C_f \in \mathcal{P}$, the idea is to use this affine weight to bound the log canonical threshold of the pair $(\mathbb{P}^2, C_f)$. The definition is as follows:

Definition II.2.2.2. *Given $\mathcal{P} \in \mathscr{P}_d$ and a one-parameter subgroup $\lambda : \mathbb{C}^\times \to SL(V)$ we define the affine weight of $\mathcal{P}$ at λ to be*

$$\omega(\mathcal{P}, \lambda) \doteq \min\{(a_x - a_z)(i + k) + (a_y - a_z)(j + l) \ : \ m_{ijkl} \neq 0\}$$

The inspiration for this definition comes from Definition 2.2 in [34] and it is justified by Lemma II.2.3.2. The notations are the same as above and, even when omitted, we will always choose coordinates $[x, y, z]$ in $\mathbb{P}^2$ so that a one-parameter subgroup λ is normalized. Then, stated in terms of $\omega(\mathcal{P}, \lambda)$, the Hilbert-Mumford criterion becomes:

Proposition II.2.2.3. *A pencil $\mathcal{P} \in \mathscr{P}_d$ is unstable (resp. not stable) if and only if there exists a one-parameter subgroup $\lambda : \mathbb{C}^\times \to SL(V)$ and a choice of coordinates in $\mathbb{P}^2$ such that*

$$\omega(\mathcal{P}, \lambda) > \frac{2d}{3}(a_x + a_y - 2a_z) \quad (\textit{resp. } \geq)$$

Proof. A pencil $\mathcal{P} \in \mathscr{P}_d$ is unstable (resp. not stable) if and only if there exists a one-parameter subgroup $\lambda : \mathbb{C}^\times \to SL(V)$ and a choice of coordinates in $\mathbb{P}^2$ satisfying that for any i, j, k and l such that $m_{ijkl} \neq 0$ (in those coordinates) we have

$$a_x(i+k) + a_y(j+l) + a_z(2d - i - j - k - l) > 0 \quad (\text{resp. } \geq 0)$$

if and only if

$$(a_x - a_z)(i+k) + (a_y - a_z)(j+l) - \frac{2d}{3}(a_x + a_y - 2a_z) > 0 \quad (\text{resp. } \geq 0)$$

$\square$

Similarly, we define an affine weight for plane curves of degree d:

Definition II.2.2.4. *Given a plane curve of degree d C_f and a one-parameter subgroup $\lambda : \mathbb{C}^\times \to SL(V)$ we define the affine weight of f at λ to be*

$$\omega(f, \lambda) \doteq \min\{(a_x - a_z)i + (a_y - a_z)j \ : \ f_{ij} \neq 0\}$$

And for curves the Hilbert-Mumford criterion becomes:

Proposition II.2.2.5. *A curve C_f is unstable (resp. not stable) if and only if there exists a one-parameter subgroup $\lambda : \mathbb{C}^\times \to SL(V)$ and a choice of coordinates in $\mathbb{P}^2$ such that*

$$\omega(f, \lambda) > \frac{d}{3}(a_x + a_y - 2a_z) \quad (\textit{resp.} \ \geq)$$

Proof. A curve C_f is unstable (resp. not stable) if and only if there exists a one-parameter subgroup $\lambda : \mathbb{C}^\times \to SL(V)$ and a choice of coordinates in $\mathbb{P}^2$ satisfying that for any i and j such that $f_{ij} \neq 0$ (in those coordinates) we have

$$a_x i + a_y j + a_z(d - i - j) > 0 \quad (\text{resp.} \ \geq 0)$$

if and only if

$$(a_x - a_z)i + (a_y - a_z)j - \frac{d}{3}(a_x + a_y - 2a_z) > 0 \quad (\text{resp.} \ \geq 0)$$

□

Given a pencil $\mathcal{P} \in \mathscr{P}_d$ and a curve $C_f \in \mathcal{P}$, it is interesting to compare the affine weights $\omega(f, \lambda)$ and $\omega(\mathcal{P}, \lambda)$ for a fixed one-parameter subgroup λ. We state and prove a series of Propositions in this direction that allow us to relate the stability of a pencil to the stability of its generators.

Proposition II.2.2.6. *Given a pencil $\mathcal{P} \in \mathscr{P}_d$ and any two (distinct) curves $C_f, C_g \in \mathcal{P}$ we have that*

$$\omega(f, \lambda) \leq \omega(f, \lambda) + \omega(g, \lambda) \leq \omega(\mathcal{P}, \lambda)$$

for all one-parameter subgroups $\lambda : \mathbb{C}^\times \to SL(V)$.

Proof. Given $\mathcal{P}$ and $\lambda : \mathbb{C}^\times \to SL(V)$, choose coordinates in $\mathbb{P}^2$ that normalize λ and choose any two curves C_f and C_g of $\mathcal{P}$ so that $\mathcal{P}$ is represented by the Plücker coordinates $m_{ijkl} = f_{ij}g_{kl} - g_{ij}f_{kl}$.

Let i, j, k and l be such that $m_{ijkl} = f_{ij}g_{kl} - g_{ij}f_{kl} \neq 0$ and

$$\omega(\mathcal{P}, \lambda) = (a_x - a_z)(i + k) + (a_y - a_z)(j + l)$$

Then either i and j are such that $f_{ij} \neq 0$ or k and l are such that $f_{kl} \neq 0$. In the first case there are two possibilities: either $g_{kl} = 0$, which implies $g_{ij} \neq 0$ and $f_{kl} \neq 0$; or $g_{kl} \neq 0$. Similarly, in the second case either $g_{ij} = 0$, which implies $g_{kl} \neq 0$ and $f_{ij} \neq 0$; or $g_{ij} \neq 0$.

In any case we have

$$\begin{aligned} (a_x - a_z)(i + k) + (a_y - a_z)(j + l) &= \big((a_x - a_z)i + (a_y - a_z)j\big) + \\ &\quad + \big((a_x - a_z)k + (a_y - a_z)l\big) \\ &\geq \omega(f, \lambda) + \omega(g, \lambda) \end{aligned}$$

□

Proposition II.2.2.7. *Given a pencil $\mathcal{P} \in \mathscr{P}_d$, a one-parameter subgroup λ of $SL(V)$ and any curve $C_f \in \mathcal{P}$, there exists a curve C_g in $\mathcal{P}$ such that*

$$\omega(\mathcal{P}, \lambda) \leq \omega(f, \lambda) + \omega(g, \lambda)$$

Proof. Fix $\lambda : \mathbb{C}^{\times} \to SL(V)$ and coordinates in $\mathbb{P}^2$ that normalize λ. Choose any two curves C_f and C_g of $\mathcal{P}$. Let i and j be such that $f_{ij} \neq 0$ and

$$\omega(f, \lambda) = (a_x - a_z)i + (a_y - a_z)j$$

Replacing g by $g' = g - \frac{g_{ij}}{f_{ij}} f$ we have $g_{ij} = 0$, hence $m_{ijkl} \neq 0$ for all k and l such that $g_{kl} \neq 0$ and it follows that

$$\omega(\mathcal{P}, \lambda) \leq \omega(f, \lambda) + \omega(g, \lambda)$$

$\square$

Corollary II.2.2.8. *Given a pencil $\mathcal{P} \in \mathscr{P}_d$, a one-parameter subgroup λ of $SL(V)$ and any curve $C_f \in \mathcal{P}$ there exists a curve C_g in $\mathcal{P}$ such that*

$$\omega(\mathcal{P}, \lambda) \leq 2 \max\{\omega(f, \lambda), \omega(g, \lambda)\}$$

Corollary II.2.2.9. *Given a pencil $\mathcal{P} \in \mathscr{P}_d$, a one-parameter subgroup λ of $SL(V)$ and any curve $C_f \in \mathcal{P}$, there exists a curve C_g in $\mathcal{P}$ such that*

$$\omega(\mathcal{P}, \lambda) = \omega(f, \lambda) + \omega(g, \lambda)$$

Corollary II.2.2.10. *If a pencil $\mathcal{P} \in \mathscr{P}_d$ has only semistable (resp. stable) members, then $\mathcal{P}$ is semistable (resp. stable).*

Corollary II.2.2.11. *If a pencil $\mathcal{P} \in \mathscr{P}_d$ contains only plane curves C_d such that the pairs $\left(\mathbb{P}^2, 3/dC_d\right)$ (resp.$\left(\mathbb{P}^2, (3/d + \varepsilon)\, C_d\right), 0 < \varepsilon << 1$) are log canonical, then $\mathcal{P}$ is semistable (resp. stable).*

Proof. As observed in [23] and [27], in this case all members of $\mathcal{P}$ are semistable (resp. stable). □

As a result of our comparison between $\omega(f,\lambda)$ and $\omega(\mathcal{P},\lambda)$ we prove Theorems II.2.2.12 and II.2.2.13 below:

Theorem II.2.2.12. *If $\mathcal{P} \in \mathscr{P}_d$ contains at worst one strictly semistable curve (and all other curves in $\mathcal{P}$ are stable), then $\mathcal{P}$ is stable.*

Proof. Given $\mathcal{P}$ as above, if all curves in $\mathcal{P}$ are stable, then $\mathcal{P}$ is stable by Corollary II.2.2.10. Otherwise, let C_f be the unique strictly semistable curve in $\mathcal{P}$. Given any one-parameter subgroup λ, by Proposition II.2.2.7 there exists a curve C_g such that

$$\frac{\omega(\mathcal{P},\lambda)}{(a_x - a_z) + (a_y - a_z)} \leq \frac{\omega(f,\lambda)}{(a_x - a_z) + (a_y - a_z)} + \frac{\omega(g,\lambda)}{(a_x - a_z) + (a_y - a_z)}$$

And because C_f (resp. C_g) is strictly semistable (resp. stable) it follows that

$$\frac{\omega(f,\lambda)}{(a_x - a_z) + (a_y - a_z)} \leq \frac{d}{3} \qquad \text{and} \qquad \frac{\omega(h,\lambda)}{(a_x - a_z) + (a_y - a_z)} < \frac{d}{3}$$

and hence

$$\frac{\omega(\mathcal{P},\lambda)}{(a_x - a_z) + (a_y - a_z)} < \frac{2d}{3}$$

That is, $\mathcal{P}$ is stable. □

Theorem II.2.2.13. *If $\mathcal{P} \in \mathscr{P}_d$ contains at worst two semistable curves C_f and C_g (and all other curves in $\mathcal{P}$ are stable), then $\mathcal{P}$ is strictly semistable if and only if there exists a one-parameter subgroup λ (and coordinates in $\mathbb{P}^2$) such that C_f and C_g are* *both non-stable with respect to this λ that is,*

$$\frac{\omega(f,\lambda)}{(a_x - a_z) + (a_y - a_z)} = \frac{d}{3} \qquad \textit{and} \qquad \frac{\omega(g,\lambda)}{(a_x - a_z) + (a_y - a_z)} = \frac{d}{3}$$

Proof. Fix $\mathcal{P}$ as above and note that $\mathcal{P}$ is semistable (Corollary II.2.2.10). First, note that if the two inequalities above hold for some λ, then $\mathcal{P}$ is strictly semistable by Proposition II.2.2.6. Thus, assume $\mathcal{P}$ is strictly semistable. Then there exists a one-parameter subgroup λ (and coordinates in $\mathbb{P}^2$) such that

$$\frac{\omega(\mathcal{P},\lambda)}{(a_x - a_z) + (a_y - a_z)} = \frac{2d}{3}$$

and, by Corollary II.2.2.8, it must exist a curve C_h in $\mathcal{P}$ such that

$$\frac{d}{3} \leq \max\left\{\frac{\omega(f,\lambda)}{(a_x - a_z) + (a_y - a_z)}, \frac{\omega(h,\lambda)}{(a_x - a_z) + (a_y - a_z)}\right\}$$

In particular, either C_f or C_h is non-stable with respect to this λ. But C_f and C_g are the only potentially non-stable curves in $\mathcal{P}$. Therefore, either

$$\frac{\omega(f,\lambda)}{(a_x - a_z) + (a_y - a_z)} \geq \frac{d}{3} \tag{II.2.2.2}$$

or $C_h = C_g$ and

$$\frac{\omega(g,\lambda)}{(a_x - a_z) + (a_y - a_z)} \geq \frac{d}{3} \tag{II.2.2.3}$$

In any case, we claim that the following two equalities hold

$$\frac{\omega(f,\lambda)}{(a_x - a_z) + (a_y - a_z)} = \frac{d}{3} \qquad \text{and} \qquad \frac{\omega(g,\lambda)}{(a_x - a_z) + (a_y - a_z)} = \frac{d}{3}$$

In fact, if $C_h = C_g$ and (II.2.2.3) holds, then

$$\frac{\omega(g,\lambda)}{(a_x - a_z) + (a_y - a_z)} = \frac{d}{3}$$

because C_g is semistable. Thus, by Proposition II.2.2.7, inequality (II.2.2.2) must be true also.

Now, if (II.2.2.2) holds, then

$$\frac{\omega(f,\lambda)}{(a_x - a_z) + (a_y - a_z)} = \frac{d}{3}$$

because C_f is semistable. Thus, by Proposition II.2.2.7, we have that

$$\frac{\omega(h,\lambda)}{(a_x - a_z) + (a_y - a_z)} \geq \frac{d}{3}$$

and, by assumption, it must be the case that $C_h = C_g$ (and (II.2.2.3) holds).

□

II.2.3 Stability and the log canonical threshold

We are now ready to describe how $\omega(\mathcal{P},\lambda)$ and $\omega(f,\lambda)$ are related to the log canonical threshold of the pair $(\mathbb{P}^2, C_f)$. We begin by proving the following:

Proposition II.2.3.1. *Given $\mathcal{P} \in \mathscr{P}_d$ and any base point p of $\mathcal{P}$, there exists a one-parameter subgroup $\lambda : \mathbb{C}^\times \to SL(V)$ (and coordinates in $\mathbb{P}^2$) such that for any curve C_f in $\mathcal{P}$ we have that*

$$\frac{(a_x - a_z) + (a_y - a_z)}{\omega(\mathcal{P},\lambda)} \leq lct_p(\mathbb{P}^2, C_f)$$

Proof. Given $\mathcal{P}$ and a base point p, we can choose coordinates in $\mathbb{P}^2$ so that $p = (0 : 0 : 1)$.

Given any $a \in \mathbb{Q} \cap (-1/2, 1]$, we can let $a_x = 1, a_y = a$ and $a_z = -1 - a$ and consider the one-parameter subgroup λ, which in these coordinates is normalized. Then

$$\frac{(a_x - a_z) + (a_y - a_z)}{\omega(\mathcal{P}, \lambda)} = \frac{3(1 + a)}{(2 + a)(i + k) + (2a + 1)(j + l)}$$

for some $0 \leq i, j, k, l \leq d$ such that $m_{ijkl} \neq 0$.

Because $f_{00} = 0$ for any curve C_f in $\mathcal{P}$, we have that $m_{00kl} = 0$ for all $0 \leq k, l \leq d$. This implies

$$\frac{3(1 + a)}{(2 + a)(i + k) + (2a + 1)(j + l)} \leq 1$$

for all i, j, k, l such that $m_{ijkl} \neq 0$.

We claim that given $a \in \mathbb{Q} \cap (-1/2, 1]$, the corresponding one-parameter subgroup λ is such that for any curve C_f in $\mathcal{P}$ we have

$$\frac{(a_x - a_z) + (a_y - a_z)}{\omega(P, \lambda)} \leq lct_p(\mathbb{P}^2, C_f)$$

By contradiction, assume there exists C_f in $\mathcal{P}$ such that

$$lct_p(\mathbb{P}^2, C_f) < \frac{(a_x - a_z) + (a_y - a_z)}{\omega(\mathcal{P}, \lambda)}$$

Write $\tilde{f}(u, v) = f(x, y, 1)$ and assign weights $\omega(u) \doteq a_x - a_z = 2 + a$ to the variable u and $\omega(v) \doteq a_y - a_z = 2a + 1$ to the variable v so that the weighted multiplicity of $\tilde{f}$ is precisely $\omega(f, \lambda)$.

Now, consider the finite morphism $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ given by $(u, v) \mapsto (u^{\omega(u)}, v^{\omega(v)})$ and let

$$\Delta \doteq (1 - \omega(u))H_u + (1 - \omega(v))H_v + c \cdot \tilde{f}(u^{\omega(u)}, v^{\omega(v)})$$

where H_u (resp. H_v) is the divisor of $u = 0$ (resp. $v = 0$) and $c \in \mathbb{Q} \cap [0, 1]$. Then

$$\varphi^*(K_{\mathbb{C}^2} + c \cdot \tilde{f}(u, v)) = K_{\mathbb{C}^2} + \Delta$$

and by Proposition 5.20 (4) in [36] we know that the pair $(\mathbb{C}^2, c \cdot \tilde{f})$ is log canonical at $(0, 0)$ if and only if the pair $(\mathbb{C}^2, \Delta)$ is log canonical at $(0, 0)$.

In particular, taking $c = \dfrac{\omega(u) + \omega(v)}{\omega(\mathcal{P}, \lambda)} > lct_p(\mathbb{P}^2, C_f) = lct_0(\mathbb{C}^2, \tilde{f})$ it follows that

$$a(E; \mathbb{C}^2, \Delta) = -1 + \omega(u) + \omega(v) - c \cdot \omega(f, \lambda) < -1$$

where E is the exceptional divisor of the blow-up of $\mathbb{C}^2$ at the origin and $a(E; \mathbb{C}^2, \Delta)$ is the corresponding discrepancy.

But the above inequality is equivalent to the inequality $\omega(\mathcal{P}, \lambda) < \omega(f, \lambda)$, which contradicts Proposition II.2.2.6. □

Next, we recall the following known result:

Lemma II.2.3.2 ([35, Proposition 8.13]). *Let C_f be any plane curve. Then*

$$\frac{\omega(f, \lambda)}{(a_x - a_z) + (a_y - a_z)} \leq \frac{1}{lct(\mathbb{P}^2, C_f)} \tag{II.2.3.1}$$

for any one-parameter subgroup $\lambda : \mathbb{C}^\times \to SL(V)$.

Proof. Fix any one-parameter subgroup $\lambda : \mathbb{C}^\times \to SL(V)$ and choose coordinates in $\mathbb{P}^2$ so that λ is normalized. There are two possibilities: either $a_y > a_z$ (hence $a_x > a_z$) or $a_y = a_z$. Let us first consider the former.

If $p \doteq (0,0,1) \notin C_f$, then $f_{00} \neq 0$, which implies $\omega(f,\lambda) = 0$ and inequality (II.2.3.1) is true.

Otherwise, we can write $\tilde{f}(u,v) = f(x,y,1)$ and assign weights $\omega(x) = a_x$ to the variable x, $\omega(y) = a_y$ to the variable y and $\omega(z) = a_z$ to the variable z. Then u has weight $a_x - a_z$, v has weight $a_y - a_z$ and we have that the weighted multiplicity of $\tilde{f}$ is precisely $\omega(f,\lambda)$.

Proposition 8.13 in [35] tells us

$$\frac{\omega(f,\lambda)}{(a_x - a_z) + (a_y - a_z)} \leq \frac{1}{lct_0(\mathbb{C}^2, \tilde{f})}$$

and the result follows from the fact that $lct(\mathbb{P}^2, C_f) \leq lct_p(\mathbb{P}^2, C_f) = lct_0(\mathbb{C}^2, \tilde{f})$.

Finally, if we are in the situation when $a_y = a_z$, then

$$\omega(f,\lambda) = \min\{(a_x - a_z)i \ ; \ f_{ij} \neq 0\}$$

and the desired inequality becomes

$$c \doteq \min\{i \ ; \ f_{ij} \neq 0\} \leq \frac{1}{lct(\mathbb{P}^2, C_f)}$$

If $c = 0$ or $c = 1$ the inequality is obvious. And if $c \geq 2$, then C_f contains a line $(x = 0)$ with multiplicity $c \geq 2$ and, again, the inequality is true. $\square$

In particular, we conclude from Corollary II.2.2.8 that:

Proposition II.2.3.3. *Given $\mathcal{P} \in \mathscr{P}_d$, and any one-parameter subgroup λ of $SL(V)$, there exists $C_f \in \mathcal{P}$ such that*

$$\frac{\omega(\mathcal{P},\lambda)}{(a_x - a_z) + (a_y - a_z)} \leq \frac{2}{lct(\mathbb{P}^2, C_f)} \tag{II.2.3.2}$$

And, as a consequence, we recover the statement from Corollary II.2.2.11:

Corollary II.2.3.4. *If $\mathcal{P} \in \mathscr{P}_d$ is a pencil such that $lct(\mathbb{P}^2, C_f) \geq 3/d$ (resp. $> 3/d$) for any curve C_f in $\mathcal{P}$, then $\mathcal{P}$ is semistable (resp. stable).*

Proposition II.2.3.1 and Lemma II.2.3.2 together with the other results obtained in this section, allow us to prove Theorems II.2.3.5 and II.2.3.6 below. Both results relate the stability of $\mathcal{P}$ and the log canonical threshold of the pair $(\mathbb{P}^2, C_f)$ for $C_f \in \mathcal{P}$.

Theorem II.2.3.5. *Let $\mathcal{P}$ be a pencil in $\mathscr{P}_d$ which contains a curve C_f such that $lct(\mathbb{P}^2, C_f) = \alpha$. If $\mathcal{P}$ is unstable (resp. not stable), then $\mathcal{P}$ contains a curve C_g such that $lct(\mathbb{P}^2, C_g) < \dfrac{3\alpha}{2d\alpha - 3}$ (resp. $\leq$).*

Proof. If $\mathcal{P}$ is unstable (resp. not stable), then by Proposition II.2.2.3 we can choose a one-parameter subgroup λ (and coordinates in $\mathbb{P}^2$) so that

$$\frac{2d}{3} < \frac{\omega(\mathcal{P}, \lambda)}{(a_x - a_z) + (a_y - a_z)} \qquad \text{(resp. } \leq\text{)}$$

By Proposition II.2.2.7, we can find a a curve C_g in $\mathcal{P}$ such that

$$\frac{\omega(\mathcal{P}, \lambda)}{(a_x - a_z) + (a_y - a_z)} \leq \frac{\omega(f, \lambda)}{(a_x - a_z) + (a_y - a_z)} + \frac{\omega(g, \lambda)}{(a_x - a_z) + (a_y - a_z)}$$

Moreover, by Lemma II.2.3.2 we have that

$$\frac{\omega(f, \lambda)}{(a_x - a_z) + (a_y - a_z)} \leq \frac{1}{lct(\mathbb{P}^2, C_f)} \quad \text{and} \quad \frac{\omega(g, \lambda)}{(a_x - a_z) + (a_y - a_z)} \leq \frac{1}{lct(\mathbb{P}^2, C_g)}$$

And, because $lct(\mathbb{P}^2, C_f) = \alpha$, combining the above inequalities we conclude that

$$\frac{2d}{3} - \frac{1}{\alpha} < \frac{1}{lct(\mathbb{P}^2, C_g)} \qquad \text{(resp. } \leq\text{)} \iff lct(\mathbb{P}^2, C_g) < \frac{3\alpha}{2d\alpha - 3} \qquad \text{(resp. } \leq\text{)}$$

$\square$

Theorem II.2.3.6. *If $\mathcal{P} \in \mathscr{P}_d$ is semistable (resp. stable), then for any curve C_f in $\mathcal{P}$ and any base point p of $\mathcal{P}$ we have $\frac{3}{2d} \leq lct_p(\mathbb{P}^2, C_f)$ (resp. $<$).*

Proof. Fix $\mathcal{P} \in \mathscr{P}_d$ and a base point p as above. Given C_f we can always find coordinates in $\mathbb{P}^2$ so that $p = (0 : 0 : 1)$ and we can choose λ as in Proposition II.2.3.1. Because $\mathcal{P}$ is semistable (resp. stable) for this λ we have that

$$\frac{3}{2d} \leq \frac{(a_x - a_z) + (a_y - a_z)}{\omega(\mathcal{P}, \lambda)} \qquad \text{(resp. } <\text{)}$$

and the result follows from Proposition II.2.3.1. $\square$

II.2.4 Stability and the multiplicity at a base point

We now relate $\omega(\mathcal{P}, \lambda)$ to the multiplicity of the generators of $\mathcal{P}$ at a base point. Our result is the following:

Theorem II.2.4.1. *Let $\mathcal{P}$ be a pencil in $\mathscr{P}_d$ with generators C_f and C_g. If there exists a base point P of $\mathcal{P}$ such that $mult_P(C_f) + mult_P(C_g) > \frac{4d}{3}$ (resp. $\geq$), then $\mathcal{P}$ is unstable (resp. not stable).*

Proof. If P is any base point of $\mathcal{P}$, we can always choose coordinates so that $P = (0 : 0 : 1)$. Let $a_x = 1, a_y = 1, a_z = -2$ and λ be the one-parameter subgroup which in these coordinates is normalized. Then $\omega(f, \lambda) = 3 \cdot \mathrm{mult}_P(C_f)$ and $\omega(g, \lambda) = 3 \cdot \mathrm{mult}_P(C_g)$ for any choice of generators of $\mathcal{P}$, say C_f and C_g. These two equalities, together with Proposition II.2.2.6, imply

$$\frac{\omega(\mathcal{P},\lambda)}{(a_x-a_z)+(a_y-a_z)} \geq 3\cdot\frac{\mathrm{mult}_P(C_f)+\mathrm{mult}_P(C_g)}{(a_x-a_z)+(a_y-a_z)}$$

And since $(a_x-a_z)+(a_y-a_z)=6$, the result then follows from the Hilbert-Mumford criterion (Proposition II.2.2.3). □

II.2.5 Stability criterion for pencils of plane sextics

We now restrict our attention to the case $d=6$, i.e. we consider pencils of plane sextics. We begin by observing that in (II.2.2.1) we can normalize the weights so that $a_x=1, a_y=a$ and $a_z=-1-a$ for some $a\in[-1/2,1]\cap\mathbb{Q}$. Then the action of $\lambda(t)\in SL(V)$ on the Plücker coordinates m_{ijkl} is given by $m_{ijkl}\mapsto t^{e_{ijkl}}m_{ijkl}$, where

$$e_{ijkl}=e_{ijkl}(a)=(2i+2k+j+l-12)+a(2j+2l+i+k-12)$$

and the Hilbert-Mumford criterion for pencils of plane sextics becomes:

Proposition II.2.5.1. *A pencil $\mathcal{P}\in\mathscr{P}_6$ is unstable (resp. not stable) if and only if there exists a rational number $a\in[-1/2,1]$ and coordinates in $\mathbb{P}^2$ such that if the pencil is represented in those coordinates by (m_{ijkl}), then $m_{ijkl}=0$ whenever $e_{ijkl}(a)\leq 0$ (resp. $e_{ijkl}(a)<0$).*

A priori, for each choice of coordinates in $\mathbb{P}^2$ one would need to test all possible values of $a\in[-1/2,1]\cap\mathbb{Q}$ to verify the stability criterion. Because the function (for a fixed $\mathcal{P}$ and a choice of coordinates)

$$\mu(\mathcal{P},\lambda)\doteq\min\{e_{ijkl}(a)\ :\ m_{ijkl}\neq 0\} \tag{II.2.5.1}$$

is piecewise linear, a key observation is that we only need to test its positivity for a finite number of critical values $a \in [-1/2, 1] \cap \mathbb{Q}$.

In other words, the conditions $e_{ijkl}(a) \leq 0$ (resp. $e_{ijkl}(a) < 0$) divide the interval $[-1/2, 1]$ into finitely many subintervals $[a_n, a_{n+1}]$ within which the truthfulness of the inequality remains constant. That is, for each interval $[a_n, a_{n+1}]$ we can find values of i, j, k and l for which the inequality $e_{ijkl}(a) \leq 0$ (resp. $e_{ijkl}(a) < 0$) remains true for all $a \in [a_n, a_{n+1}]$.

To find these intervals we proceed as follows. For computational reasons we first let $r = i + k$ and $s = j + l$. Then, for each possible pair (r, s) in the set

$$\{(r, s) \in \{0, 1, \ldots, 12\} \times \{0, 1, \ldots, 12\} \ : \ r + s \leq 12\},$$

we test whether we can solve the inequality $2r + s - 12 + a(2s + r - 12) \leq 0$ (resp. < 0) for the variable a imposing the restriction $a \in [-1/2, 1]$.

There are $\binom{14}{2}$ such pairs so the use of a computer program comes in handy. In Table II.2.1 below we present our results.

	Values of r and s	**Interval**
Unstability	$r = 4$ and $s = 5, \ldots, 8$	$[-1/2]$
	$r = 0, 1, 2, 3$ or 4 and $s = 0, \ldots, 8 - r$	$[-1/2, 1]$
	$r = 0$ and $s = 9$	$[-1/2, 1/2]$
	$r = 1$ and $s = 8$	$[-1/2, 2/5]$
	$r = 0$ and $s = 10$; $r = 2$ and $s = 7$	$[-1/2, 1/4]$
	$r = 1$ and $s = 9$	$[-1/2, 1/7]$
	$r = 0$ and $s = 11$	$[-1/2, 1/10]$
	$r = 0, 1, 2$ or 3 and $s = 12 - 2r$	$[-1/2, 0]$
	$r = 1$ and $s = 11$	$[-1/2, -1/11]$
	$r = 2$ and $s = 9$	$[-1/2, -1/8]$
	$r = 2$ and $s = 10$; $r = 3$ and $s = 7$	$[-1/2, -1/5]$
	$r = 3$ and $s = 8$	$[-1/2, -2/7]$
	$r = 3$ and $s = 9$	$[-1/2, -1/3]$
	$r = 5$ and $s = 0$	$[-2/7, 1]$
	$r = 5$ and $s = 1$	$[-1/5, 1]$
	$r = 5$ and $s = 2$; $r = 6$ and $s = 0$	$[0, 1]$
	$r = 6$ and $s = 1$	$[1/4, 1]$
	$r = 7$ and $s = 0$	$[2/5, 1]$
	$r = 5, 6, 7$ or 8 and $s = 8 - r$	$[1]$

	Values of r and s	**Interval**
Non stability	$r = 0, 1, 2$ or 3 and $s = 0, \ldots, 7 - r$	$[-1/2, 1]$
	$r = 0, 1, 2$ or 3 and $s = 8 - r$	$[-1/2, 1)$
	$r = 4$ and $s = 0, \ldots, 3$	$(-1/2, 1]$
	$r = 0$ and $s = 9$	$[-1/2, 1/2)$
	$r = 1$ and $s = 8$	$[-1/2, 2/5)$
	$r = 0$ and $s = 10$; $r = 2$ and $s = 7$	$[-1/2, 1/4)$
	$r = 1$ and $s = 9$	$[-1/2, 1/7)$
	$r = 0$ and $s = 11$	$[-1/2, 1/10)$
	$r = 0, 1, 2$ or 3 and $s = 12 - 2r$	$[-1/2, 0)$
	$r = 1$ and $s = 11$	$[-1/2, -1/11)$
	$r = 2$ and $s = 9$	$[-1/2, -1/8)$
	$r = 2$ and $s = 10$; $r = 3$ and $s = 7$	$[-1/2, -1/5)$
	$r = 3$ and $s = 8$	$[-1/2, -2/7)$
	$r = 3$ and $s = 9$	$[-1/2, -1/3)$
	$r = 5$ and $s = 0$	$(-2/7, 1]$
	$r = 5$ and $s = 1$	$(-1/5, 1]$
	$r = 5$ and $s = 2$; $r = 6$ and $s = 0$	$(0, 1]$
	$r = 6$ and $s = 1$	$(1/4, 1]$
	$r = 7$ and $s = 0$	$(2/5, 1]$

Table II.2.1: Intervals for unstability and non stability

In summary, the intervals we find are given by Lemmas II.2.5.2 and II.2.5.3 below:

Lemma II.2.5.2. *The condition $e_{ijkl}(a) \leq 0$ divides the interval $[-1/2, 1]$ into finitely many subintervals and in order to obtain minimal conditions for unstability, it suffices considering only the following six distinct subintervals:*

$$(-1/3, -2/7), (-2/7, -1/5), (-1/11, 0), (1/7, 1/4), (1/4, 2/5), (1/2, 1)$$

Lemma II.2.5.3. *The condition $e_{ijkl}(a) < 0$ divides the interval $[-1/2, 1]$ into finitely many subintervals and the subintervals that give (distinct) minimal conditions for non-stability are such that it suffices taking $a \in \{-1/2, -2/7, -1/5, 0, 1/4, 2/5, 1\}$.*

In particular, we can restate the criteria for unstability (resp. non-stability) as in Theorem II.2.5.4 (resp. Theorem II.2.5.5):

Theorem II.2.5.4. *A pencil $\mathcal{P} \in \mathscr{P}_6$ is unstable if and only if there exist coordinates in $\mathbb{P}^2$ so that if the pencil is represented in those coordinates by (m_{ijkl}), then $m_{ijkl} = 0$ whenever the (appropriate) values of i, j, k and l satisfy either one of the following conditions:*

1. $(2i + 2k + j + l - 12) - 13/42(2j + 2l + i + k - 12) \leq 0$
2. $(2i + 2k + j + l - 12) - 8/35(2j + 2l + i + k - 12) \leq 0$
3. $(2i + 2k + j + l - 12) - 1/12(2j + 2l + i + k - 12) \leq 0$
4. $(2i + 2k + j + l - 12) + 3/14(2j + 2l + i + k - 12) \leq 0$

5. $(2i+2k+j+l-12)+3/10(2j+2l+i+k-12) \leq 0$

6. $(2i+2k+j+l-12)+3/4(2j+2l+i+k-12) \leq 0$

Theorem II.2.5.5. *A pencil $\mathcal{P} \in \mathscr{P}_6$ is not stable if and only if there exist coordinates in $\mathbb{P}^2$ so that if the pencil is represented in those coordinates by (m_{ijkl}), then $m_{ijkl} = 0$ whenever the (appropriate) values of i, j, k and l satisfy either one of the following conditions:*

1. $(2i+2k+j+l-12)-1/2(2j+2l+i+k-12) < 0$

2. $(2i+2k+j+l-12)-2/7(2j+2l+i+k-12) < 0$

3. $(2i+2k+j+l-12)-1/5(2j+2l+i+k-12) < 0$

4. $(2i+2k+j+l-12) < 0$

5. $(2i+2k+j+l-12)+1/4(2j+2l+i+k-12) < 0$

6. $(2i+2k+j+l-12)+2/5(2j+2l+i+k-12) < 0$

7. $(2i+2k+j+l-12)+(2j+2l+i+k-12) < 0$

Now, in order to know what is the set of values i, j, k and l for which the Plücker coordinates m_{ijkl} vanish in Theorems II.2.5.4 and II.2.5.5 above, it is convenient to express these values in terms of the pairs (r, s). For each pair (r, s) we let

$$M_{rs} \doteq \{m_{ijkl} \ : \ i+k=r \text{ and } j+l=s\}$$

and we obtain the following:

Theorem II.2.5.6. *A pencil $\mathcal{P} \in \mathscr{P}_6$ is unstable if and only if there exist coordinates in $\mathbb{P}^2$ so that if the pencil is represented in those coordinates by (m_{ijkl}), then either*

1. *$M_{rs} = \{0\}$ for all the pairs (r, s) in the list below*

$$\begin{array}{cccccccccc}
(0,0) & (0,1) & (0,2) & (0,3) & (0,4) & (0,5) & (0,6) & (0,7) & (0,8) & (0,9) \\
(0,10) & (0,11) & (0,12) & (1,0) & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(1,7) & (1,8) & (1,9) & (1,10) & (1,11) & (2,0) & (2,1) & (2,2) & (2,3) & (2,4) \\
(2,5) & (2,6) & (2,7) & (2,8) & (2,9) & (2,10) & (3,0) & (3,1) & (3,2) & (3,3) \\
(3,4) & (3,5) & (3,6) & (3,7) & (3,8) & (4,0) & (4,1) & (4,2) & (4,3) & (4,4)
\end{array}$$

   and a number $a \in (-1/3, -2/7)$ will exhibit $\mathcal{P}$ as unstable; or

2. *$M_{rs} = \{0\}$ for all the pairs (r, s) in the list below:*

$$\begin{array}{cccccccccc}
(0,0) & (0,1) & (0,2) & (0,3) & (0,4) & (0,5) & (0,6) & (0,7) & (0,8) & (0,9) \\
(0,10) & (0,11) & (0,12) & (1,0) & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(1,7) & (1,8) & (1,9) & (1,10) & (1,11) & (2,0) & (2,1) & (2,2) & (2,3) & (2,4) \\
(2,5) & (2,6) & (2,7) & (2,8) & (2,9) & (2,10) & (3,0) & (3,1) & (3,2) & (3,3) \\
(3,4) & (3,5) & (3,6) & (3,7) & (4,0) & (4,1) & (4,2) & (4,3) & (4,4) & (5,0)
\end{array}$$

   and a number $a \in (-2/7, -1/5)$ will exhibit $\mathcal{P}$ as unstable; or

3. $M_{rs} = \{0\}$ *for all the pairs* (r, s) *in the list below*

$$
\begin{array}{cccccccccc}
(0,0) & (0,1) & (0,2) & (0,3) & (0,4) & (0,5) & (0,6) & (0,7) & (0,8) & (0,9) \\
(0,10) & (0,11) & (0,12) & (1,0) & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(1,7) & (1,8) & (1,9) & (1,10) & (2,0) & (2,1) & (2,2) & (2,3) & (2,4) & (2,5) \\
(2,6) & (2,7) & (2,8) & (3,0) & (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,0) & (4,1) & (4,2) & (4,3) & (4,4) & (5,0) & (5,1) & & &
\end{array}
$$

and a number $a \in (-1/11, 0)$ *will exhibit* $\mathcal{P}$ *as unstable; or*

4. $M_{rs} = \{0\}$ *for all the pairs* (r, s) *in the list below*

$$
\begin{array}{cccccccccc}
(0,0) & (0,1) & (0,2) & (0,3) & (0,4) & (0,5) & (0,6) & (0,7) & (0,8) & (0,9) \\
(0,10) & (1,0) & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) & (1,7) & (1,8) \\
(2,0) & (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) & (2,7) & (3,0) & (3,1) \\
(3,2) & (3,3) & (3,4) & (3,5) & (4,0) & (4,1) & (4,2) & (4,3) & (4,4) & (5,0) \\
(5,1) & (5,2) & (6,0) & & & & & & &
\end{array}
$$

and a number $a \in (1/7, 1/4)$ *will exhibit* $\mathcal{P}$ *as unstable; or*

5. $M_{rs} = \{0\}$ *for all the pairs* (r, s) *in the list below*

$$
\begin{array}{cccccccccc}
(0,0) & (0,1) & (0,2) & (0,3) & (0,4) & (0,5) & (0,6) & (0,7) & (0,8) & (0,9) \\
(1,0) & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) & (1,7) & (1,8) & (2,0) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) & (3,0) & (3,1) & (3,2) & (3,3) \\
(3,4) & (3,5) & (4,0) & (4,1) & (4,2) & (4,3) & (4,4) & (5,0) & (5,1) & (5,2) \\
(6,0) & (6,1) & & & & & & & &
\end{array}
$$

and a number $a \in (1/4, 2/5)$ *will exhibit* $\mathcal{P}$ *as unstable; or*

6. $M_{rs} = \{0\}$ *for all the pairs* (r, s) *in the list below*

$$
\begin{array}{cccccccccc}
(0,0) & (0,1) & (0,2) & (0,3) & (0,4) & (0,5) & (0,6) & (0,7) & (0,8) & (1,0) \\
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) & (1,7) & (2,0) & (2,1) & (2,2) \\
(2,3) & (2,4) & (2,5) & (2,6) & (3,0) & (3,1) & (3,2) & (3,3) & (3,4) & (3,5) \\
(4,0) & (4,1) & (4,2) & (4,3) & (4,4) & (5,0) & (5,1) & (5,2) & (6,0) & (6,1) \\
(7,0) & & & & & & & & &
\end{array}
$$

and a number $a \in (1/2, 1)$ *will exhibit* $\mathcal{P}$ *as unstable.*

Theorem II.2.5.7. *A pencil* $\mathcal{P} \in \mathscr{P}_6$ *is not stable if and only if there exist coordinates in* $\mathbb{P}^2$ *so that if the pencil is represented in those coordinates by* (m_{ijkl})*, then either*

1. $M_{rs} = \{0\}$ *for all the pairs* (r, s) *in the list below*

$$
\begin{array}{cccccccccc}
(0,0) & (0,1) & (0,2) & (0,3) & (0,4) & (0,5) & (0,6) & (0,7) & (0,8) & (0,9) \\
(0,10) & (0,11) & (0,12) & (1,0) & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(1,7) & (1,8) & (1,9) & (1,10) & (1,11) & (2,0) & (2,1) & (2,3) & (2,4) & (2,5) \\
(2,6) & (2,7) & (2,8) & (2,9) & (2,10) & (3,0) & (3,1) & (3,2) & (3,3) & (3,4) \\
(3,5) & (3,6) & (3,7) & (3,8) & (3,9) & & & & &
\end{array}
$$

and $a = -1/2$ *will exhibit* $\mathcal{P}$ *as not stable; or*

2. $M_{rs} = \{0\}$ *for all the pairs* (r, s) *in the list below*

$$\begin{array}{cccccccccc} (0,0) & (0,1) & (0,2) & (0,3) & (0,4) & (0,5) & (0,6) & (0,7) & (0,8) & (0,9) \\ (0,10) & (0,11) & (0,12) & (1,0) & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (1,7) & (1,8) & (1,9) & (1,10) & (1,11) & (2,0) & (2,1) & (2,3) & (2,4) & (2,5) \\ (2,6) & (2,7) & (2,8) & (2,9) & (2,10) & (3,0) & (3,1) & (3,2) & (3,3) & (3,4) \\ (3,5) & (3,6) & (3,7) & (4,0) & (4,1) & (4,2) & (4,3) & & & \end{array}$$

and $a = -2/7$ *will exhibit* $\mathcal{P}$ *as not stable; or*

3. $M_{rs} = \{0\}$ *for all the pairs* (r, s) *in the list below*

$$\begin{array}{cccccccccc} (0,0) & (0,1) & (0,2) & (0,3) & (0,4) & (0,5) & (0,6) & (0,7) & (0,8) & (0,9) \\ (0,10) & (0,11) & (0,12) & (1,0) & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (1,7) & (1,8) & (1,9) & (1,10) & (1,11) & (2,0) & (2,1) & (2,3) & (2,4) & (2,5) \\ (2,6) & (2,7) & (2,8) & (2,9) & (3,0) & (3,1) & (3,2) & (3,3) & (3,4) & (3,5) \\ (3,6) & (4,0) & (4,1) & (4,2) & (4,3) & (5,0) & & & & \end{array}$$

and $a = -1/5$ *will exhibit* $\mathcal{P}$ *as not stable; or*

4. $M_{rs} = \{0\}$ *for all the pairs* (r, s) *in the list below*

$$\begin{array}{cccccccccc} (0,0) & (0,1) & (0,2) & (0,3) & (0,4) & (0,5) & (0,6) & (0,7) & (0,8) & (0,9) \\ (0,10) & (0,11) & (1,0) & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) & (1,7) \\ (1,8) & (1,9) & (2,0) & (2,1) & (2,3) & (2,4) & (2,5) & (2,6) & (2,7) & (3,0) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (4,0) & (4,1) & (4,2) & (4,3) & (5,0) \\ (5,1) & & & & & & & & & \end{array}$$

and $a = 0$ *will exhibit* $\mathcal{P}$ *as not stable; or*

5. $M_{rs} = \{0\}$ *for all the pairs* (r, s) *in the list below*

$$(0,0) \quad (0,1) \quad (0,2) \quad (0,3) \quad (0,4) \quad (0,5) \quad (0,6) \quad (0,7) \quad (0,8) \quad (0,9)$$
$$(1,0) \quad (1,1) \quad (1,2) \quad (1,3) \quad (1,4) \quad (1,5) \quad (1,6) \quad (1,7) \quad (1,8) \quad (2,0)$$
$$(2,1) \quad (2,3) \quad (2,4) \quad (2,5) \quad (2,6) \quad (3,0) \quad (3,1) \quad (3,2) \quad (3,3) \quad (3,4)$$
$$(3,5) \quad (4,0) \quad (4,1) \quad (4,2) \quad (4,3) \quad (5,0) \quad (5,1) \quad (5,2) \quad (6,0)$$

and $a = 1/4$ *will exhibit* $\mathcal{P}$ *as not stable; or*

6. $M_{rs} = \{0\}$ *for all the pairs* (r, s) *in the list below*

$$(0,0) \quad (0,1) \quad (0,2) \quad (0,3) \quad (0,4) \quad (0,5) \quad (0,6) \quad (0,7) \quad (0,8) \quad (0,9)$$
$$(1,0) \quad (1,1) \quad (1,2) \quad (1,3) \quad (1,4) \quad (1,5) \quad (1,6) \quad (1,7) \quad (2,0) \quad (2,1)$$
$$(2,3) \quad (2,4) \quad (2,5) \quad (2,6) \quad (3,0) \quad (3,1) \quad (3,2) \quad (3,3) \quad (3,4) \quad (3,5)$$
$$(4,0) \quad (4,1) \quad (4,2) \quad (4,3) \quad (5,0) \quad (5,1) \quad (5,2) \quad (6,0) \quad (6,1)$$

and $a = 2/5$ *will exhibit* $\mathcal{P}$ *as not stable; or*

7. $M_{rs} = \{0\}$ *for all the pairs* (r, s) *in the list below*

$$(0,0) \quad (0,1) \quad (0,2) \quad (0,3) \quad (0,4) \quad (0,5) \quad (0,6) \quad (0,7) \quad (1,0) \quad (1,1)$$
$$(1,2) \quad (1,3) \quad (1,4) \quad (1,5) \quad (1,6) \quad (2,0) \quad (2,1) \quad (2,3) \quad (2,4) \quad (2,5)$$
$$(3,0) \quad (3,1) \quad (3,2) \quad (3,3) \quad (3,4) \quad (4,0) \quad (4,1) \quad (4,2) \quad (4,3) \quad (5,0)$$
$$(5,1) \quad (5,2) \quad (6,0) \quad (6,1) \quad (7,0)$$

and $a = 1$ *will exhibit* $\mathcal{P}$ *as not stable.*

II.2.5.1 A geometric description

We have completely characterized the stability of a pencil $\mathcal{P} \in \mathscr{P}_6$ in terms of its Plücker coordinates (m_{ijkl}). But now we want to understand which are the geometric properties unstable and non-stable pencils have. More precisely, we want to translate the stability criteria into equations for the generators of the pencil.

Throughout this section, given an unstable (resp. not stable) pencil $\mathcal{P} \in \mathscr{P}_6$ we choose coordinates $[x, y, z]$ in $\mathbb{P}^2$ as in Theorem II.2.5.4 (resp. II.2.5.5) and generators C_f and C_g having defining polynomials (in these coordinates) $f = \sum f_{ij} x^i y^j z^{6-i-j}$ and $g = \sum g_{ij} x^i y^j z^{6-i-j}$. Then, the idea is that each vanishing condition $m_{ijkl} = 0$ translates into the vanishing of the coefficients of some pair $C_{f'}$ and $C_{g'}$ of generators (not necessarily the original pair).

To illustrate what kind of computations are involved in this process we prove Theorem II.2.5.8 below. We use the notation $\langle m_1, \ldots, m_n \rangle$ to denote the subspace of homogeneous polynomials of degree six in the variables x, y and z which is generated by the monomials m_i. Whereas $\rangle m_1, \ldots, m_n \langle$ denotes the subspace of those polynomials which are generated by all the monomials which are different from the m_i.

Theorem II.2.5.8. *A pencil $\mathcal{P} \in \mathscr{P}_6$ satisfies the vanishing conditions in case* 1 *of Theorem II.2.5.7 if and only if there exist coordinates in $\mathbb{P}^2$ and generators C_f and C_g of $\mathcal{P}$ such that either*

Case 1 $f \in \langle x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$ *and g is arbitrary*

Case 2 $f \in \langle x^3z^3, x^3yz^3, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

$$\textit{and } g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6\langle$$

Case 3 f *and* $g \in \langle x^iy^jz^{6-i-j}\rangle$, *where* $2 \leq i \leq 6, 0 \leq j \leq 6$ *and* $i + j \leq 6$

Proof. Let us assume $\mathcal{P}$ is not stable and that its Plücker coordinates (m_{ijkl}) must vanish for all i, j, k and l satisfying $i + k = r$ and $j + l = s$ for all the pairs (r, s) in case 1 of Theorem II.2.5.7. Using the relations $m_{ijkl} = -m_{klij}$ and $m_{ijij} = 0$ we can compute the minimal set of values $\{i, j, k, l\}$ (in order) so that the m_{ijkl} vanish.

In other words, we find all integers i, j, k and l subject to the restrictions

(i) $0 \leq i, j, k, l \leq 6$,

(ii) $i + j \leq 6$,

(iii) $k + l \leq 6$, and

(iv) $(i < k) \vee (i = k \wedge j < l)$

satisfying the inequality

$$(2i + 2k + j + l - 12) - 1/2(2j + 2l + i + k - 12) < 0$$

All possible solutions $\{i, j, k, l\}$ (in order) are:

$\{0,0,0,1\},\{0,0,0,2\},\{0,0,0,3\},\{0,0,0,4\},\{0,0,0,5\},\{0,0,0,6\},\{0,0,1,0\},$

$\{0,0,1,1\},\{0,0,1,2\},\{0,0,1,3\},\{0,0,1,4\},\{0,0,1,5\},\{0,0,2,0\},\{0,0,2,1\},$

$\{0,0,2,2\},\{0,0,2,3\},\{0,0,2,4\},\{0,0,3,0\},\{0,0,3,1\},\{0,0,3,2\},\{0,0,3,3\},$

$\{0,1,0,2\},\{0,1,0,3\},\{0,1,0,4\},\{0,1,0,5\},\{0,1,0,6\},\{0,1,1,0\},\{0,1,1,1\},$

$\{0,1,1,2\},\{0,1,1,3\},\{0,1,1,4\},\{0,1,1,5\},\{0,1,2,0\},\{0,1,2,1\},\{0,1,2,2\},$

$\{0,1,2,3\},\{0,1,2,4\},\{0,1,3,0\},\{0,1,3,1\},\{0,1,3,2\},\{0,1,3,3\},\{0,2,0,3\},$

$\{0,2,0,4\},\{0,2,0,5\},\{0,2,0,6\},\{0,2,1,0\},\{0,2,1,1\},\{0,2,1,2\},\{0,2,1,3\},$

$\{0,2,1,4\},\{0,2,1,5\},\{0,2,2,0\},\{0,2,2,1\},\{0,2,2,2\},\{0,2,2,3\},\{0,2,2,4\},$

$\{0,2,3,0\},\{0,2,3,1\},\{0,2,3,2\},\{0,2,3,3\},\{0,3,0,4\},\{0,3,0,5\},\{0,3,0,6\},$

$\{0,3,1,0\},\{0,3,1,1\},\{0,3,1,2\},\{0,3,1,3\},\{0,3,1,4\},\{0,3,1,5\},\{0,3,2,0\},$

$\{0,3,2,1\},\{0,3,2,2\},\{0,3,2,3\},\{0,3,2,4\},\{0,3,3,0\},\{0,3,3,1\},\{0,3,3,2\},$

$\{0,3,3,3\},\{0,4,0,5\},\{0,4,0,6\},\{0,4,1,0\},\{0,4,1,1\},\{0,4,1,2\},\{0,4,1,3\},$

$\{0,4,1,4\},\{0,4,1,5\},\{0,4,2,0\},\{0,4,2,1\},\{0,4,2,2\},\{0,4,2,3\},\{0,4,2,4\},$

$\{0,4,3,0\},\{0,4,3,1\},\{0,4,3,2\},\{0,4,3,3\},\{0,5,0,6\},\{0,5,1,0\},\{0,5,1,1\},$

$\{0,5,1,2\},\{0,5,1,3\},\{0,5,1,4\},\{0,5,1,5\},\{0,5,2,0\},\{0,5,2,1\},\{0,5,2,2\},$

$\{0,5,2,3\},\{0,5,2,4\},\{0,5,3,0\},\{0,5,3,1\},\{0,5,3,2\},\{0,5,3,3\},\{0,6,1,0\},$

$\{0,6,1,1\},\{0,6,1,2\},\{0,6,1,3\},\{0,6,1,4\},\{0,6,1,5\},\{0,6,2,0\},\{0,6,2,1\},$

$\{0,6,2,2\},\{0,6,2,3\},\{0,6,2,4\},\{0,6,3,0\},\{0,6,3,1\},\{0,6,3,2\},\{0,6,3,3\},$

$\{1,0,1,1\},\{1,0,1,2\},\{1,0,1,3\},\{1,0,1,4\},\{1,0,1,5\},\{1,0,2,0\},\{1,0,2,1\},$

$\{1,0,2,2\},\{1,0,2,3\},\{1,0,2,4\},\{1,1,1,2\},\{1,1,1,3\},\{1,1,1,4\},\{1,1,1,5\},$

$\{1,1,2,0\},\{1,1,2,1\},\{1,1,2,2\},\{1,1,2,3\},\{1,1,2,4\},\{1,2,1,3\},\{1,2,1,4\},$

$\{1,2,1,5\},\{1,2,2,0\},\{1,2,2,1\},\{1,2,2,2\},\{1,2,2,3\},\{1,2,2,4\},\{1,3,1,4\},$

$\{1,3,1,5\},\{1,3,2,0\},\{1,3,2,1\},\{1,3,2,2\},\{1,3,2,3\},\{1,3,2,4\},\{1,4,1,5\},$

$\{1,4,2,0\},\{1,4,2,1\},\{1,4,2,2\},\{1,4,2,3\},\{1,4,2,4\},\{1,5,2,0\},\{1,5,2,1\},$

$\{1,5,2,2\},\{1,5,2,3\},\{1,5,2,4\}$

The question then is how to determine which coefficients in the defining polynomials of the generators need to vanish.

Note that we have introduced an ordering on the Plücker coordinates coming from the restrictions on i, j, k and l. So, the first step is to look at the equation $m_{ijkl} = 0$ for the first term $\{i, j, k, l\}$ in the list above, namely we look at the equation $m_{0001} = 0$. It follows that either

(1) $f_{00} = g_{00} = 0$ or

(2) $g_{00} \neq 0$ or

(3) $f_{00} \neq 0$

Moreover, if (2) (or (3) by symmetry) holds, then taking $f' = f - \dfrac{f_{00}}{g_{00}} g$ we can assume $f_{00} = 0$ and we must have $f_{01} = 0$.

The next step then is, at each of the cases above, to look at the next vanishing condition $m_{0002} = 0$ coming from the second term $\{i, j, k, l\}$ in the list. Again there are three possibilities: Either $f_{00} = g_{00} = 0$ or $g_{02} \neq 0$ or $f_{02} \neq 0$.

We proceed in this manner until there are no more equations $m_{ijkl} = 0$ to solve.

In fact our list tells us that m_{00kl} vanish for all (appropriate) $0 \leq k \leq 3$ and $0 \leq l \leq 6$. Thus, our algorithm tells us that if we are in the situation of case (2), then one of the generators belongs to $\rangle x^k j^l z^{6-k-l} \langle$ for all kl such that $m_{00kl} = 0$. And, by symmetry, we reach the same conclusion if (3) holds. A similar reasoning applies to the next set of vanishing conditions $m_{01kl} = 0$ and so on.

It is important to note, however, that at each step, when solving the equations $m_{ijkl} = 0$ we have to take into account whether there are or there are not previous conditions on the coefficients f_{ij}, g_{ij}, f_{kl} and g_{kl}.

Following the sketched algorithm we obtain the desired geometric description of the pencil $\mathcal{P}$. □

Note that the same algorithm outlined above in the proof of Theorem II.2.5.8 can be applied more generally whenever $\mathcal{P}$ is unstable (resp. not stable) and satisfies one of the vanishing conditions in anyone of the cases in Theorem II.2.5.6 (resp. II.2.5.7). However, the computations involved are very lengthy and the assistance of a computer is needed. And even the corresponding statements as in Theorem II.2.5.8 require several pages to be presented.

The complete geometric description of the stability conditions in terms of equations for the generators is presented in Appendix B. And instead of exhibiting these tiresome results we will present next (without proofs) those which are essential in the study of Halphen pencils of index two (Chapter II.3) and we will mostly focus on **proper** pencils:

Definition II.2.5.9. *A pencil $\mathcal{P} \in \mathscr{P}_6$ is called proper if any two curves on it intersect properly, meaning its base locus is zero dimensional, i.e. it consists of a finite number of points.*

II.2.5.1.1 Equations associated to nonstability

Theorem II.2.5.10. *Let $\mathcal{P} \in \mathscr{P}_6$ be a proper pencil which contains a curve of the form $3L + C$, where L is a line and C is a cubic (possibly reducible). Then $\mathcal{P}$ is not stable if and only if there exist coordinates in $\mathbb{P}^2$ and generators C_f and C_g of $\mathcal{P}$ such that:*

(a) $f \in \langle x^3z^3, x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$ *with* $f_{30} \neq 0$ *and* g *satisfies*

- *(a1)* $g_{00} = \ldots = g_{05} = 0, g_{10} = \ldots = g_{13} = 0, g_{20} = g_{21} = 0$ *or*
- *(a2)* $g_{00} = \ldots = g_{04} = 0, g_{10} = \ldots = g_{13} = 0, g_{20} = g_{21} = g_{22} = g_{31} = g_{40} = 0$

(b) $f \in \langle x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$ *with* $f_{31} \neq 0$ *and* g *satisfies*

- *(b1)* $g_{00} = \ldots = g_{05} = 0, g_{10} = g_{11} = g_{12} = 0$ *or*
- *(b2)* $g_{00} = \ldots = g_{04} = 0, g_{10} = g_{11} = g_{12} = g_{20} = 0$ *or*

(b3) $g_{00} = \ldots = g_{03} = 0, g_{10} = g_{11} = g_{12} = g_{20} = g_{21} = g_{30} = 0$

(c) $f \in \langle x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$ *with* $f_{32} \neq 0$ *and either*

(c1) g *satisfies* $g_{00} = \ldots = g_{03} = 0, g_{10} = g_{11} = 0$ *or*

(c2) $m_{ijkl} = 0$ *for* i, j, k, l *(in order) in the list below*

$$\{0, 3, 4, 0\}, \{1, 2, 4, 0\}, \{2, 1, 4, 0\}, \{3, 0, 4, 0\}$$

and g *satisfies* $g_{00} = g_{01} = g_{02} = g_{10} = g_{11} = g_{20} = 0$

(d) $f \in \langle x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$ *with* $f_{33} \neq 0$ *and either*

(d1) g *satisfies* $g_{00} = \ldots = g_{03} = 0, g_{10} = 0$ *or*

(d2) $m_{ijkl} = 0$ *for* i, j, k, l *(in order) in the list below*

$$\{0, 3, 4, 0\}, \{1, 1, 4, 0\}$$

and g *satisfies* $g_{00} = g_{01} = g_{02} = g_{10} = 0$ *or*

(d3) $m_{ijkl} = 0$ *for* i, j, k, l *(in order) in the list below*

$$\{0, 2, 4, 0\}, \{0, 2, 4, 1\}, \{0, 2, 5, 0\}, \{0, 3, 4, 0\}, \{1, 1, 4, 0\}, \{1, 1, 4, 1\}, \{1, 1, 5, 0\},$$
$$\{2, 0, 4, 0\}, \{2, 1, 4, 0\}, \{3, 0, 4, 0\}$$

and g *satisfies* $g_{00} = g_{01} = g_{10} = 0$

Theorem II.2.5.11. *Let $\mathcal{P} \in \mathscr{P}_6$ be a proper pencil which contains a curve of the form $2L + Q$, where L is a line and Q is a quartic (possibly reducible). Then $\mathcal{P}$ is not stable if and only if there exist coordinates in $\mathbb{P}^2$ and generators C_f and C_g of $\mathcal{P}$ such that:*

(a) *$f \in \langle x^2z^4, x^2yz^3, x^2y^2z^2, x^2y^3z, x^2y^4, x^iy^jz^{6-i-j}\rangle$, with $3 \leq i \leq 6$, $0 \leq j \leq 6, i+j \leq 6$ plus $f_{20} \neq 0$ and $g \in \langle y^6, xy^5, x^2y^4, x^3y^3, x^4y^2, x^5y, x^6\rangle$ (in particular, C_g is unstable)*

(b) *$f \in \langle x^2yz^3, x^2y^2z^2, x^2y^3z, x^2y^4, x^iy^jz^{6-i-j}\rangle$, with $3 \leq i \leq 6, 0 \leq j \leq 6, i+j \leq 6$ plus $f_{21} \neq 0$ and g satisfies*

  (b1) *$g_{00} = \ldots = g_{05} = 0, g_{10} = \ldots = g_{14} = 0, g_{20} = g_{21} = g_{22} = g_{30} = g_{31} = 0$ or*

  (b2) *$g_{00} = \ldots = g_{04} = 0, g_{10} = \ldots = g_{13} = 0, g_{20} = g_{21} = g_{22} = g_{30} = g_{31} = g_{40} = 0$*

  in particular, C_g is unstable.

(c) *$f \in \langle x^2y^2z^2, x^2y^3z, x^2y^4, x^iy^jz^{6-i-j}\rangle$, with $3 \leq i \leq 6, 0 \leq j \leq 6, i+j \leq 6$ plus $f_{22} \neq 0$ and either*

  (c1) *g satisfies $g_{00} = \ldots = g_{05} = 0, g_{10} = \ldots = g_{13} = 0, g_{20} = g_{21} = 0$ or*

  (c2) *$f_{30} = 0$ and g satisfies $g_{00} = \ldots = g_{04} = 0, g_{10} = \ldots = g_{13} = 0, g_{20} = g_{21} = g_{22} = g_{30} = 0$ or* *(c3)* $m_{ijkl} = 0$ *for* i, j, k, l *(in order) in the list below*

$$\{0,4,3,0\}, \{1,3,3,0\}, \{3,0,3,1\}, \{3,0,4,0\}$$

and g *satisfies* $g_{00} = \ldots = g_{03} = 0, g_{10} = g_{11} = g_{12} = g_{20} = g_{21} = g_{22} = g_{30} = 0$

In particular, $(0:0:1)$ *has multiplicity* ≥ 3 *in* C_g.

(d) $f \in \langle x^2y^3z, x^2y^4, x^iy^jz^{6-i-j}\rangle$*, with* $3 \leq i \leq 6, 0 \leq j \leq 6, i+j \leq 6$ *plus* $f_{23} \neq 0$ *and either*

(d1) $m_{ijkl} = 0$ *for* i, j, k, l *(in order) in the list below*

$$\{0,5,3,0\}, \{1,3,3,0\}, \{2,1,3,0\}$$

and g *satisfies* $g_{00} = \ldots = g_{04} = 0, g_{10} = g_{11} = g_{12} = g_{20} = 0$ *or*

(d2) $m_{ijkl} = 0$ *for* i, j, k, l *(in order) in the list below*

$$\{0,4,3,0\}, \{0,4,3,1\}, \{0,5,3,0\}, \{1,3,3,0\}, \{2,1,3,0\}, \{2,1,3,1\}, \{2,2,3,0\}$$

and g *satisfies* $g_{00} = \ldots = g_{03} = 0, g_{10} = g_{11} = g_{12} = g_{20} = 0$ *or*

(d3) $m_{ijkl} = 0$ *for* i, j, k, l *(in order) in the list below*

$$\{0,3,3,0\}, \{0,3,3,1\}, \{0,3,4,0\}, \{0,4,3,0\}, \{1,2,3,0\}, \{1,2,3,1\}, \{1,2,4,0\},$$
$$\{1,3,3,0\}, \{2,1,3,0\}, \{2,1,3,1\}, \{2,1,4,0\}, \{2,2,3,0\}, \{3,0,3,1\}, \{3,0,4,0\}$$

and g *satisfies* $g_{00} = g_{01} = g_{02} = g_{10} = g_{11} = g_{20} = 0$

In particular, $(0:0:1)$ *has multiplicity* ≥ 3 *in* C_g*.*

(e) $f \in \langle x^2y^4, x^iy^jz^{6-i-j}\rangle$, *with* $3 \leq i \leq 6, 0 \leq j \leq 6, i+j \leq 6$ *plus* $f_{24} \neq 0$ *and either*

(e1) $m_{ijkl} = 0$ *for* i, j, k, l *(in order) in the list below*

$$\{0,6,3,0\}, \{1,3,3,0\}, \{2,0,3,0\}$$

and g *satisfies* $g_{00} = \ldots = g_{05} = 0, g_{10} = g_{11} = g_{12} = 0$ *or*

(e2) $m_{ijkl} = 0$ *for* i, j, k, l *(in order) in the list below*

$$\{0,4,3,0\}, \{0,4,3,1\}, \{0,5,3,0\}, \{1,2,3,0\}, \{1,2,3,1\}, \{1,3,3,0\}, \{2,0,3,0\},$$
$$\{2,0,3,1\}, \{2,1,3,0\}$$

and g *satisfies* $g_{00} = \ldots = g_{03} = 0, g_{10} = g_{11} = 0$ *or*

(e3) $m_{ijkl} = 0$ *for* i, j, k, l *(in order) in the list below*

$$\{0,3,3,0\}, \{0,3,3,1\}, \{0,3,3,2\}, \{0,3,4,0\}, \{0,4,3,0\}, \{0,4,3,1\}, \{0,5,3,0\},$$
$$\{1,2,3,0\}, \{1,2,3,1\}, \{1,2,4,0\}, \{1,3,3,0\}, \{2,0,3,0\}, \{2,0,3,1\}, \{2,0,3,2\},$$
$$\{2,1,3,0\}, \{2,1,3,1\}, \{2,2,3,0\}$$

and g *satisfies* $g_{00} = g_{01} = g_{02} = g_{10} = g_{11} = 0$ *or* *(e4)* $m_{ijkl} = 0$ *for* i, j, k, l *(in order) in the list below*

$$\{0,2,3,0\},\{0,2,3,1\},\{0,2,3,2\},\{0,2,4,0\},\{0,2,4,1\},\{0,2,5,0\},\{0,3,3,0\},$$
$$\{0,3,3,1\},\{0,3,4,0\},\{0,4,3,0\},\{1,1,3,0\},\{1,1,3,1\},\{1,1,3,2\},\{1,1,4,0\},$$
$$\{1,1,4,1\},\{1,1,5,0\},\{1,2,3,0\},\{1,2,3,1\},\{1,2,4,0\},\{1,3,3,0\},\{2,0,3,0\},$$
$$\{2,0,3,1\},\{2,0,3,2\},\{2,0,4,0\},\{2,0,4,1\},\{2,0,5,0\},\{2,1,3,0\},\{2,1,3,1\},$$
$$\{2,1,4,0\},\{2,2,3,0\},\{3,0,3,1\},\{3,0,4,0\}$$

and g *satisfies* $g_{00} = g_{01} = g_{10} = 0$

II.2.5.1.2 Equations associated to unstability

Theorem II.2.5.12. *A pencil* $\mathcal{P} \in \mathscr{P}_6$ *will satisfy the vanishing conditions in case 1 of Theorem II.2.5.6 if and only if we can find coordinates in* $\mathbb{P}^2$ *and generators* C_f *and* C_g *of* $\mathcal{P}$ *such that*

Case 1 $f \in \langle x^5z, x^5y, x^6\rangle$ *and* g *is arbitrary*

Case 2 $f \in \langle x^4y^2, x^5z, x^5y, x^6\rangle$ *and* $g \in \rangle z^6, yz^5, y^2z^4\langle$

Case 3 $f \in \langle x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$ *and* $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3\langle$

Case 4 $f \in \langle x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$ *and* $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2\langle$

Case 5 $f \in \langle x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z\langle$

Case 6 $f \in \langle x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3\langle$

Case 7 $f \in \langle x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2\langle$

Case 8 $f \in \langle x^3z^3, x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2, xy^4z, xy^5\langle$

Case 9 $f \in \langle x^2y^4, x^3z^3, x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

and

$g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2, xy^4z, x^2z^4\langle$

Theorem II.2.5.13. *Let $\mathcal{P} \in \mathscr{P}_6$ be a proper pencil which contains a curve of the form $4L + Q$, where L is a line and Q is a conic (possibly reducible). Then $\mathcal{P}$ is unstable if and only if there exist coordinates in $\mathbb{P}^2$ and generators C_f and C_g of $\mathcal{P}$ such that:*

(a) $f \in \langle x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$ *plus* $f_{40} \neq 0$ *and either* g *satisfies*

(a1) $g_{00} = \ldots = g_{04} = 0$ *or*

(a2) $g_{00} = \ldots = g_{03} = 0, g_{10} = g_{11} = g_{12} = g_{20} = 0$ *(in particular,* $(0:0:1)$ *has multiplicity* ≥ 3 *in* C_g*.).*

(b) $f \in \langle x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$ *plus* $f_{41} \neq 0$ *and* g *satisfies* $g_{00} = \ldots = g_{03} = 0$.

(c) $f \in \langle x^4y^2, x^5z, x^5y, x^6\rangle$ *plus* $f_{42} \neq 0$ *and* g *satisfies* $g_{00} = g_{01} = g_{02} = 0$.

Theorem II.2.5.14. *Let* $\mathcal{P} \in \mathscr{P}_6$ *be a proper pencil which contains a curve of the form* $3L + C$*, where* L *is a line and* C *is a cubic (possibly reducible). Then* $\mathcal{P}$ *is unstable if and only if there exist coordinates in* $\mathbb{P}^2$ *and generators* C_f *and* C_g *of* $\mathcal{P}$ *such that:*

(a) $f \in \langle x^3z^3, x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$ *plus* $f_{30} \neq 0$ *and* g *satisfies*

$$g_{00} = \ldots = g_{05} = 0, g_{10} = \ldots = g_{14} = 0, g_{20} = g_{21} = g_{22} = 0$$

In particular, $(0:0:1)$ *has multiplicity* ≥ 3 *in* C_g.

(b) $f \in \langle x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$ *plus* $f_{31} \neq 0$ *and* g *satisfies*

$$g_{00} = \ldots = g_{04} = 0, g_{10} = \ldots = g_{13} = 0, g_{20} = g_{21} = 0$$

In particular, $(0:0:1)$ *has multiplicity* ≥ 3 *in* C_g.

(c) $f \in \langle x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$ *plus* $f_{32} \neq 0$ *and* g *satisfies*

$$g_{00} = \ldots = g_{04} = 0, g_{10} = g_{11} = g_{12} = 0$$

(d) $f \in \langle x^3y^2z, x^3y^3, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$ *plus* $f_{32} \neq 0$ *and* g *satisfies*

$$g_{00} = \ldots = g_{03} = 0, g_{10} = g_{11} = g_{12} = g_{20} = 0$$

In particular, $(0:0:1)$ *has multiplicity* ≥ 3 *in* C_g.

(e) $f \in \langle x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$ *plus* $f_{33} \neq 0$ *and* g *satisfies*

$$g_{00} = \ldots = g_{04} = 0, g_{10} = g_{11} = 0$$

(f) $f \in \langle x^3y^3, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$ *plus* $f_{33} \neq 0$ *and* g *satisfies*

$$g_{00} = g_{01} = g_{02} = g_{10} = g_{11} = 0$$

Theorem II.2.5.15. *Let* $\mathcal{P} \in \mathscr{P}_6$ *be a proper pencil which contains a curve of the form* $2L+Q$*, where* L *is a line and* Q *is a quartic (possibly reducible). If* $\mathcal{P}$ *is unstable then there exist coordinates in* $\mathbb{P}^2$ *and generators* C_f *and* C_g *of* $\mathcal{P}$ *such that:*

$$f \in \langle x^2z^4, x^2yz^3, x^2y^2z^2, x^2y^3z, x^2y^4, x^iy^jz^{6-i-j}\rangle$$

with $3 \leq i \leq 6, 0 \leq j \leq 6, i+j \leq 6$ *plus* $f_{2j} \neq 0$ *for some* $j = 0, \ldots, 4$ *and* $(0:0:1)$ *has multiplicity* ≥ 3 *in* C_g*.*

Chapter II.3

Stability of Halphen pencils of index two

Building in the results we obtained in [54] and [55], in this chapter (see also [56]) we completely describe the stability of Halphen pencils of index two as points in the Grassmannian $Gr(2, 28)$ (see Sections II.2.2 and II.2.5). These are classical geometric objects that were first introduced by the French mathematician Georges Henri Halphen in 1882 [24]. They consist of pencils of plane curves of degree six with exactly nine base points (possibly infinitely near) of multiplicity two (Definition II.3.1.4). Inspired by [40], we provide a complete and geometric characterization of their stability in terms of the types of singular fibers appearing in the associated rational elliptic surfaces.

In general, a Halphen pencil (of index m) corresponds to a rational surface Y that admits a genus one fibration $f: Y \to \mathbb{P}^1$ with exactly one multiple fiber of multiplicity m (see Section II.3.1 below). Here we are interested in the case $m = 2$.

Surprisingly, Halphen pencils have appeared in [7] in the solution of a problem in Diophantine geometry and a generalization to higher dimensions has been considered in [12] and [13]. Other possible applications include the study of certain $K3$ surfaces [1], [57] and the construction of: F-theory compactifications [29],[30], and discrete Painlevé equations [49].

II.3.1 Halphen pencils and rational elliptic surfaces

In this section we present a brief discussion on rational surfaces Y that admit a genus one fibration $f: Y \to C$. These will be called **rational elliptic surfaces** and we will always make the assumption that Y is relatively minimal. Recall that, by definition, a rational surface is a surface (smooth and complete) Y which is birationally equivalent to $\mathbb{P}^2$.

We begin by proving two general results about rational elliptic surfaces. We first show any rational elliptic surface must be fibered over $\mathbb{P}^1$ and we compute its Hodge numbers:

Proposition II.3.1.1. *If a rational surface Y admits a genus one fibration $f: Y \to C$ (over $k = \mathbb{C}$), then $C \simeq \mathbb{P}^1$ and some classical invariants are encoded by the Hodge diamond:*

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
0 & & 1+9 & & 0 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

Proof. Note that because Y is rational we have that $k(Y) \simeq k(x_0, x_1, x_2) = k(\mathbb{P}^2)$, which is a pure transcendental extension of $k = \mathbb{C}$ that contains k. Now, the surjective map f induces an inclusion of function fields $k(C) \subset k(Y)$, so by Luroth's theorem we conclude that $C \simeq \mathbb{P}^1$ (see e.g. Hartshorne page 303, Example 2.5.5).

Next we compute the Hodge numbers of Y. We first note that Y is smooth and complete, hence $h^{0,0} = h^{2,2} = 1$. Now, the Hodge numbers $h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2}$ and $h^{2,0} = h^{0,2}$ are birational invariants and since Y is rational, it follows that $h^{1,0}(Y) = h^{1,0}(\mathbb{P}^2) = 0$ and $h^{2,0}(Y) = h^{2,0}(\mathbb{P}^2) = 0$. Finally, we have that $K_Y^2 = 0$, so it follows from Noether's formula that $e(Y) = 12$, hence $h^{1,1} = 10$. Here $e(Y)$ denotes the topological Euler characteristic of Y. $\square$

In fact we will see in Proposition II.3.1.9 that f is given by $|-mK_Y|$, where $m = d_X$ is the index of f (Definition I.2.0.4). In particular, if a rational surface admits a genus one fibration, then such structure is unique. We will also see that m agrees with the multiplicity of the unique multiple fiber (if there is no section) or equals 1 (if there is a section). And, moreover, Y can be obtained as a nine-point blow-up of $\mathbb{P}^2$. The latter is actually a consequence of the more general result stated next:

Lemma II.3.1.2 ([25, Lemma 4.2])**.** *Let Y be a smooth rational surface having an irreducible curve F that is linearly equivalent to a positive multiple of $-K_Y$. If $9 - K_Y^2 \geq 2$, then Y is obtained by consecutively blowing-up precisely $9 - K_Y^2$ (possibly infinitely near) points of $\mathbb{P}^2$.*

Whereas the fact that any rational elliptic surface of index $m > 1$ has a unique multiple fiber of multiplicity m follows from:

Lemma II.3.1.3 ([16, Proposition 5.61,(iii)])**.** *Let $f : Y \to \mathbb{P}^1$ be a rational elliptic surface, then f has at most one multiple fiber.*

Proof. By the canonical bundle formula we have

$$\omega_Y = f^*\big(\mathcal{O}_{\mathbb{P}^1}(-1)\big) \otimes \mathcal{O}_X\left(\sum_p (m(p)-1)Y_p\right)$$

where $m(p)$ denotes the multiplicity of the fiber Y_p at a point $p \in \mathbb{P}^1$. Thus, for any $n \in \mathbb{N}$ it follows that

$$nK_Y \sim n \cdot \left(-1 + \sum_p \frac{m(p)-1}{m(p)}\right) F$$

where F is any fiber of f. Now, because no multiple of K_Y can be effective, we must have $\displaystyle\sum_p \frac{m(p)-1}{m(p)} < 1$ and the latter implies $m(p) = 1$ except for at most one point $p \in \mathbb{P}^1$. □

Now, before we can state and prove Proposition II.3.1.9 we need to first introduce some definitions and notations.

Definition II.3.1.4. *A* ***Halphen pencil of index*** m *is a pencil of plane curves of degree* $3m$ *through nine (possibly infinitely near) singular points* $P_1, \ldots, P_9$ *of multiplicity* m*.*

Remark II.3.1.5. *Note that the generic fiber of a Halphen pencil (of index* m*) is a genus one curve.*

Definition II.3.1.6. *An irreducible plane curve of degree* $3m$*, with nine points (possibly infinitely near) of multiplicity* m *and of genus one is called a* ***Halphen curve of index*** m*.*

The next two Lemmas tell us rational elliptic surfaces and Halphen pencils are closely related:

Lemma II.3.1.7 ([5])**.** *If* $f : Y \to \mathbb{P}^1$ *is a rational elliptic surface of index* m*, then the image of the generic fiber of* $|-mK_Y|$ *under ANY birational morphism* $Y \to \mathbb{P}^2$ *is a Halphen curve of index* m*.*

Lemma II.3.1.8 ([5])**.** *If* $\mathcal{C}$ *is a Halphen curve of index* $m \geq 2$*, then the blow-up of its nine singular points (of multiplicity* m*) is a rational elliptic surface of index* m*.*

In fact there is a one-to-one correspondence between Halphen pencils (of index m) and rational elliptic fibrations (of index m):

Proposition II.3.1.9 ([16, Theorem 5.6.1], [17, Main Theorem 2.1])**.** *Let* $f : Y \to \mathbb{P}^1$ *be a rational elliptic surface of index* m *and let* F *be a choice of a fiber of* f*, then* *there exists a birational map $\pi : Y \to \mathbb{P}^2$ so that $f \circ \pi^{-1}$ is a Halphen pencil (of index m) and, moreover, $B \doteq \pi(F)$ is a plane curve of degree $3m$:*

$$\begin{tikzcd} F \subset Y \arrow[r, "\pi"] \arrow[d, "f"'] & \mathbb{P}^2 \supset B \arrow[dl] \\ \mathbb{P}^1 & \end{tikzcd}$$

Conversely, given a Halphen pencil of index m, taking the minimal resolution of its base points we get a rational elliptic surface of index m.

Proof. Note that if we are given a Halphen pencil of index m, then unwinding the definitions, it is easy to see that if we consider the minimal resolution of the base points in the pencil, then we will get a rational elliptic surface of index m. So, we will only prove the forward statement is true.

Let $f : Y \to \mathbb{P}^1$ be rational elliptic surface of index m. We will first show $f : Y \to \mathbb{P}^1$ is given by $|-mK_Y|$ and then we will show Y is a nine-point blow-up of $\mathbb{P}^2$.

Since Y does not have a section, by Lemma II.3.1.3 we know that Y has a unique multiple fiber mE. Note that we are assuming the index of $f : Y \to \mathbb{P}^1$ is m. Now, by the canonical bundle formula for elliptic surfaces,

$$K_Y \sim f^*\mathcal{O}_{\mathbb{P}^1}(-1) + (m-1)E$$

Note that $-1 = \chi(\mathcal{O}_Y) - 2 \cdot \chi(\mathcal{O}_{\mathbb{P}^1})$. Moreover, $f^*\mathcal{O}_{\mathbb{P}^1}(1) \sim mE$. Thus, $K_Y \sim -E$ and therefore, the anti-pluricanonical map given by $|-mK_Y|$ is isomorphic to the original fibration f up to projective equivalence of the base.

Now, because $-mK_Y$ is nef, for any non-singular rational curve C on Y we have that

$$C \cdot F = -mC \cdot K_Y = m \cdot (C^2 + 2) \geq 0$$

where F is a fiber of f. This implies $C^2 \geq -2$, which further implies Y can be blown down to $\mathbb{P}^2$ or $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$.

The latter statement follows from Castelnuovo's contraction theorem (see e.g. [4]): If Y has no (-1)-curves, then Y is minimal, hence it is isomorphic to either $\mathbb{P}^2$ or $\mathbb{F}_n$ for some $n \neq 1$ and, since $C^2 \geq -2$ for any smooth rational curve C on Y, it must be the case that $n = 0$ or $n = 2$. On the other hand, contracting all $(-1)-$curves on Y we obtain a birational morphism $\phi : Y \to \tilde{Y}$, where $\tilde{Y}$ is a minimal rational surface. That is, $\tilde{Y}$ is isomorphic to either $\mathbb{P}^2$ or $\mathbb{F}_n$ for some $n \neq 1$. Now, n cannot be grater than 2, because if we look at the proper transform of the negative section under ϕ, then such curve satisfies $C^2 \leq -n < -2$, a contradiction. Thus, $n \leq 2$.

Note that $\mathbb{F}_1$ is not minimal and if either $Y \simeq \mathbb{P}^2$ or $\tilde{Y} \simeq \mathbb{P}^2$ we are done, there is nothing more to prove. Therefore, it suffices to assume there is a birational morphism $\phi : Y \to \mathbb{F}_n$ for $n = 0$ or $n = 2$. Now, because ϕ factors through $\psi : Bl_x\mathbb{F}_n \doteq \overline{\mathbb{F}_n} \to \mathbb{F}_n$, where x is any indeterminacy point for ϕ^{-1}, we get the desired birational map $\pi : Y \to \mathbb{P}^2$ by constructing a map $\overline{\mathbb{F}_n} \to \mathbb{P}^2$:

$$\begin{tikzcd}
Y \arrow[r] \arrow[rr, bend left, "\phi"] \arrow[dr, dashed, "\pi"'] & \overline{\mathbb{F}_n} \arrow[r, "\psi"] \arrow[d] & \mathbb{F}_n \\
& \mathbb{P}^2 &
\end{tikzcd}$$

Explicitly, if $n = 2$ we can go from $\overline{\mathbb{F}_2}$ to $\mathbb{P}^2$ by blowing down $\psi^{-1}(L)$ and $\psi^{-1}(C)$, where L is the ruling through x and C is the (-2) section. Note that in this case we are taking x disjoint from the negative section. Otherwise, we would obtain a smooth rational curve having self-intersection smaller than -2, which we know it can't happen. Finally, if $n = 0$ we can go from $\overline{\mathbb{F}_0}$ to $\mathbb{P}^2$ by blowing down the proper transform of the two rulings at x. That is, in any case we see we can actually construct the desired birational map $\pi : Y \to \mathbb{P}^2$, which is a nine-point blow-up $(9 = \rho(Y) - \rho(\mathbb{P}^2) = h^{1,1}(Y) - h^{1,1}(\mathbb{P}^2))$.

Moreover, by construction $\pi(F) \sim \pi(-mK_Y) \sim 3mH \in \mathcal{O}_{\mathbb{P}^2}(3m)$. That is, the image of any fiber under π is a plane curve of degree $3m$. Further, we have that

$$-E \sim K_Y \sim \pi^* K_{\mathbb{P}^2} + E_1 + \ldots + E_9 \sim -3L + \sum E_i$$

where the E_i are the exceptional divisors over the 9 base points $P_i \in \mathbb{P}^2$ and L is the proper transform of a line in $\mathbb{P}^2$ (say $H \in \mathcal{O}_{\mathbb{P}^2}(1)$).

Note also that by construction any exceptional curve R on Y satisfies $R \cdot F = m$ and $\pi(F)$ has nine m-multiple base points. □

Corollary II.3.1.10. *Any Halphen pencil of index m contains exactly one cubic of multiplicity m, which corresponds to the unique multiple fiber in the associated rational elliptic surface (Lemma II.3.1.3). Since we are working in characteristic zero, the cubic corresponds to a fiber of type I_n for some $n \leq 9$* [16, Proposition 5.1.8]. *If none of the base points are singular points of the cubic, then we can further restrict to $n \leq 3$.*

Remark II.3.1.11. *By Lemma I.2.1.6 we know that any rational elliptic surface*

$Y \to \mathbb{P}^1$ of index m admits a multisection of degree m. Any such multisection is mapped by the blowing-down $\pi : Y \to \mathbb{P}^2$ to either a base point of the corresponding Halphen pencil or to a curve which, outside the base points, intersects the generic member of the pencil at exactly m points.

Finally, as a consequence of Lemma I.2.1.4, one proves the following result, which allows us to describe which are the possible types of singular fibers appearing in a rational elliptic surface (Proposition II.3.1.13 below).

Theorem II.3.1.12 ([16, Corollary 5.4.7]). *Let $J \to \mathbb{P}^1$ be a rational elliptic surface with section. Given $m \geq 1$ and a closed point $p \in \mathbb{P}^1$ such that J_p is of type $I_n, 0 \leq n \leq 9$, there exists a rational elliptic surface $Y \to \mathbb{P}^1$ of index m with unique multiple fiber $Y_p = m\overline{Y}_p$ satisfying $\overline{Y}_p \simeq J_p$. Moreover, $[Y]$ is an element of order m in $H^1(\mathbb{P}^1, \mathcal{J})$, the group of isomorphism classes of torsors over the generic fiber J_η.*

Proof. Let $J \to \mathbb{P}^1$ be a rational elliptic surface with section and let us denote the function field of $\mathbb{P}^1$ by k. Fix $m \geq 1$ and choose $p \in \mathbb{P}^1$ so that J_p is either smooth or of multiplicative type. Then there exists a non-trivial element ε_m of order m in the group J_p^0, the connected component of $J_p^\#$ intersecting the section[1]. Translation by this element defines an automorphism σ_{ε_m} of $J_p^\#$ of order m, so that the action of $J_p^\#$ on the quotient $J_p^\#/(\sigma_{\varepsilon_m})$ has a stabilizer $\simeq \mathbb{Z}/m\mathbb{Z}$. Now, such quotient defines a unique isomorphism class in $WC(J_p^\#/k)$ and, therefore, a unique isomorphism class in $WC(J_p(\bar{p})/k_p)$ via base change. Denote such class by $[Y(p)]$. Then, by Lemma

[1]In fact $J_p^\#[m] \simeq (\mathbb{Z}/m\mathbb{Z})^{b_1(J_p)}$.

I.2.1.4 , there is a unique class $[Y]$ in $H^1(\mathbb{P}^1, \mathcal{J})$ that is uniquely determined by the (only) non-zero local invariant $\tau_p([Y]) = [Y(p)]$. By construction, $f : Y \to \mathbb{P}^1$ has a unique multiple fiber $Y_p = m\overline{Y_p}$ (with $\overline{Y}_p \simeq J_p$) and the order of $[Y]$ in $H^1(\mathbb{P}^1, \mathcal{J})$ is m. Finally, to see Y is rational note that any surface X in $H^1(\mathbb{P}^1, \mathcal{J})$ has the same Hodge diamond as J; thus, $p_g(Y) = q(Y) = 0$. Moreover, by the canonical bundle formula (applied to $Y \to \mathbb{P}^1$) we have that $-mK_Y \sim F$, where F is any fiber of f; hence, the plurigenera $h^0(Y, nK_Y)$ vanish for all $n \geq 1$. $\square$

As already mentioned in the text, it is well known that the associated Jacobian fibration $J \to \mathbb{P}^1$ (see Section I.2.1) of a rational elliptic surface $Y \to \mathbb{P}^1$ is also a rational elliptic fibration, but with a section [16, Proposition 5.6.1 (ii)]. Moreover, the so called Shioda-Tate formula applied to the Jacobian fibration $J \to \mathbb{P}^1$ implies that the possible singular fibers occurring on J (hence on Y) can have at most 9 irreducible components. In particular, following Kodaira's classification, if F is a singular fiber of a rational elliptic surface $Y \to \mathbb{P}^1$, then F is of type I_n for $n \leq 9, II, III, IV, I_n^*$ for $n \leq 4, II^*, III^*$ or IV^*. In fact, given any integer $m > 1$ any type in this list can be realized by some rational elliptic surface $Y \to \mathbb{P}^1$ of index m. More precisely,

Proposition II.3.1.13. *If Y_p is a non-multiple fiber of a rational elliptic fibration $Y \to \mathbb{P}^1$ of index m, then $b_2(Y_p) \leq 9$ and any Kodaira type satisfying this condition can be realized.*

Proof. It is known [16, Corollary 5.6.6],[25, Proposition 6.1] that all such types can be realized as a (non-multiple) fiber of a rational elliptic fibration $f' : Y' \to \mathbb{P}^1$ with a section.[2] Therefore, it is sufficient to prove that given a rational elliptic surface $f' : Y' \to \mathbb{P}^1$ with a section one can construct a rational elliptic surface $f : Y \to \mathbb{P}^1$ of index m whose Jacobian fibration is precisely $f' : Y' \to \mathbb{P}^1$. This is the content of Theorem II.3.1.12 above. Note that f and f' have the same type of (non-multiple) singular fibers (see e.g. [16, Theorem 5.3.1]). □

II.3.1.1 The curves in a Halphen pencil

We will now establish a dictionary between the curves in a Halphen pencil and the fibers in the corresponding rational elliptic surface. In particular, we will provide a description of the singularities of a plane curve in a Halphen pencil. But first we need to introduce some notations and deduce some equations.

We will fix a Halphen pencil of index m and we will denote it by $\mathcal{P}$. The corresponding rational elliptic surface will be denoted by $f : Y \to \mathbb{P}^1$ and $\pi : Y \to \mathbb{P}^2$ will denote the blow-up at the nine base points of $\mathcal{P}$.

If F is any (non-multiple) fiber of Y we will denote by B the corresponding plane curve of degree $3m$, i.e. $\pi(F)$. Further, mC will denote the unique multiple cubic of $\mathcal{P}$ and mE will denote the unique multiple fiber of f.

Because $-K_Y$ is nef, every smooth rational curve R on Y has self-intersection $R^2 \geq -2$ (adjunction formula). This implies we can write the set of base points of $\mathcal{P}$

[2] In fact all such types, except types I_7 and I_3^*, can be realized as a (non-multiple) fiber of an extremal rational elliptic fibration with a section [16, Theorem 5.6.2].

as in [8, Section 2]:

$$\{P_1^{(1)}, \ldots, P_1^{(a_1)}, \ldots, P_k^{(1)}, \ldots, P_k^{(a_k)}\} \tag{II.3.1.1}$$

where $a_1 + \ldots + a_k = 9$, $P_j^{(1)}$ are points in $\mathbb{P}^2$ and $P_j^{(i+1)}$ is infinitely near to the previous point $P_j^{(i)}$ (of order 1) (see Definition II.3.1.14 below).

Definition II.3.1.14. *Given X a smooth algebraic variety of dimension $n > 1$ and $x = x^{(1)} \in X$ a closed point, consider $\pi : \tilde{X} \doteq Bl_x X \to X$ the blow-up of X at x. A closed point $x^{(2)} \in \tilde{X}$ lying in $E = \pi^{-1}(x)$ is called an infinitely near point to x of order* 1*. Inductively, an infinitely near point to x of order k is an infinitely near point (of order* 1*) to an infinitely near point (to x) of order $k-1$.*

Moreover, if C is smooth and we choose a flex point as the origin for the group law $\oplus$ on C, then [8]:

$$a_1 P_1^{(1)} \oplus \ldots \oplus a_k P_k^{(1)} = \varepsilon_m$$

where ε_m is a torsion point of order m in C (w.r.t $\oplus$).

Expressing the base points of $\mathcal{P}$ as in (II.3.1.1) is the same as saying that each exceptional curve $E_j \doteq \pi^{-1}(P_j^{(1)})$ consists of a chain of (-2) curves of length $(a_j - 1)$ with one more (-1) curve at the end of the chain. The latter a multisection of degree m.

Thus, whenever we write

$$F = \overline{F} + d_1^{(1)} E_1^{(1)} + \ldots + d_1^{(a_1-1)} E_1^{(a-1)} + \ldots + d_k^{(1)} E_k^{(1)} + \ldots + d_k^{(a_k-1)} E_k^{(a_k-1)} \tag{II.3.1.2}$$

where $\overline{F}$ denotes the strict transform of B under π and each $E_j^{(i)}$ is the

π-exceptional divisor over the base point $P_j^{(i)}$; we have the following (dual) picture for the components of E_j appearing in the fiber F:

Figure II.3.1: Chains of exceptional rational curves appearing in F

Because the chains E_j are disjoint from each other, it follows that:

Lemma II.3.1.15. *If we color the nodes of the dual graph of F corresponding to the components coming from B in blue and the nodes corresponding to the exceptional components $d_j^{(i)}E_j^{(i)}$ in black, then every black node is connected to at most two other black nodes.*

This simple observation has some interesting consequences like Propositions II.3.1.16 and II.3.1.17 below. In Appendix A we also use Lemma II.3.1.15 repeatedly in order to characterize which curves B can yield a fiber of type II^*, III^* or IV^* when $m = 2$.

Proposition II.3.1.16. *If F is of type II^*, III^* or IV^*, then $B \doteq \pi(F)$ cannot be reduced.*

Proof. If B were reduced, then coloring the dual graph of F as in Lemma II.3.1.15 we would obtain a black node which is connected to more than two black nodes. $\square$

Proposition II.3.1.17. *If F is of type II^*, then $M_B \geq 3$, where M_B denotes the largest multiplicity of a component of B.*

Proof. Again, we look at the dual graph of F. Assuming $M_B < 3$ contradicts Lemma II.3.1.15. □

Writing F as in (II.3.1.2) we can further deduce Equation (II.3.1.3) below, which computes the number of components of F.

Proposition II.3.1.18. *If n_F and n_B denote the number of components of F and B, respectively, then*

$$n_F = n_B + \sum_{j=1}^{k}(a_j - 1) - n_{E\setminus C} = n_B + \sum_{j=1}^{k} a_j - k - n_{E\setminus C} = n_B + 9 - k - n_{E\setminus C} \quad \text{(II.3.1.3)}$$

where $n_{E\setminus C}$ denotes the difference between the number of components of E and the number of components of C.

The type of the multiple fiber mE imposes restrictions on the numbers $n_{E\setminus C}$ appearing in Equation II.3.1.3 above. For instance, whenever $m > 1$ we have that

Lemma II.3.1.19. *If F is of type IV^*, then $n_{E\setminus C} \in \{0, 1, 2\}$ and if F is of type III^* (resp. II^*), then $n_{E\setminus C} \in \{0, 1\}$ (resp. $n_{E\setminus C} = 0$).*

Proof. If $m > 1$ and F is of type IV^*, III^* or II^*, then the classification in [43] tells us the unique multiple fiber mE of Y can be realized as the strict transform of mC. If F is of type IV^*, then E is of type I_0, I_1, I_2 or I_3. Whereas if F is of type III^* (resp. II^*), then E is of type I_0, I_1 or I_2 (resp. I_0 or I_1). □

Remark II.3.1.20. *If $m = 1$ and B is any given curve in $\mathcal{P}$, then we can always take the other generator of $\mathcal{P}$ to be a smooth cubic. In particular, we can always assume that $n_{E\setminus C} = 0$ in Equation (II.3.1.3).*

We can also write

$$K_Y = \pi^* K_{\mathbb{P}^2} + b_1^{(1)} E_1^{(1)} + \ldots + b_1^{(a_1)} E_1^{(a_1)} + \ldots + b_k^{(1)} E_k^{(1)} + \ldots + b_k^{(a_k)} E_k^{(a_k)}$$

and

$$\pi^* B = \overline{F} + c_1^{(1)} E_1^{(1)} + \ldots + c_1^{(a_1)} E_1^{(a_1)} + \ldots + c_k^{(1)} E_k^{(1)} + \ldots + c_k^{(a_k)} E_k^{(a_k)}$$

and we know how to compute each of the multiplicities $b_j^{(i)} \doteq b_j^{(i)}(B), c_j^{(i)} \doteq c_j^{(i)}(B)$ and $d_j^{(i)} \doteq d_j^{(i)}(B)$ rather explicitly.

For any base point $P_j^{(1)}$, the induced pencil on the surface obtained by blowing-up $P_j^{(1)}$ is

$$(\pi_j^{(1)})^* \mathcal{P} - m E_j^{(1)}$$

where $\pi_j^{(1)}$ is the blow-up map. In particular, given any curve B of $\mathcal{P}$, the induced member is

$$B_j^{(1)} + (m_{P_j^{(1)}}(B) - m) E_j^{(1)}$$

where $B_j^{(1)}$ is the strict transform of B under $\pi_j^{(1)}$ and $m_{P_j^{(1)}}(B)$ denotes the multiplicity of the point $P_j^{(1)}$ on the curve B.

In other words, $d_j^{(1)} = m_{P_j^{(1)}}(B) - m$ and, more generally,

$$d_j^{(i)} = d_j^{(i-1)} + m_{P_j^{(i)}}(B) - m$$

where $m_{P_j^{(i)}}(B)$ denotes the multiplicity of the point $P_j^{(i)}$ on the strict transform of the curve B under the blow-up of $P_j^{(1)}, \ldots, P_j^{(i-1)}$.

On the other hand, we also know that $c_j^{(1)} = m_{p_j^{(1)}}(B)$ and

$$c_j^{(i)} = c_j^{(i-1)} + m_{P_j^{(i)}}(B) = m_{P_j^{(1)}}(B) + \ldots + m_{P_j^{(i)}}(B) \qquad \text{(II.3.1.4)}$$

Thus,

$$d_j^{(i)} = c_j^{(i-1)} + m_{P_j^{(i)}}(B) - i \cdot m = c_j^{(i)} - i \cdot m = m_{P_j^{(1)}}(B) + \ldots + m_{P_j^{(i)}}(B) - i \cdot m \quad \text{(II.3.1.5)}$$

In particular,

$$d_j^{(i)} \leq i \cdot (m_{P_j^{(1)}}(B) - m) \leq i \cdot 2m \quad \text{(II.3.1.6)}$$

And the condition $d_j^{(a_j)} = 0$ implies

$$m_{P_j^{(1)}}(B) + \ldots + m_{P_j^{(a_j)}}(B) = a_j \cdot m \quad \text{(II.3.1.7)}$$

Therefore, whenever C is smooth at the base point $P_j^{(1)}$, using Noether's formula [19] we obtain

$$I_{P_j^{(1)}}(B, C) = a_j \cdot m \quad \text{(II.3.1.8)}$$

where $I_{P_j^{(1)}}(B, C)$ denotes the intersection multiplicity of B and C at the point $P_j^{(1)}$.

Lastly, we have $b_j^{(i)} = i$ for all $j = 1, \ldots k$ and $i = 1, \ldots, a_j$.

II.3.1.1.1 The (unique) multiple cubic

The cubic C is smooth at every base point of $\mathcal{P}$ if and only if π restricts to an isomorphism $E \simeq C$. This implies any $\pi-$exceptional curve must be either a multisection or a component of F.

We also prove a partial converse of this statement:

Lemma II.3.1.21. *For any index m and any type of fiber we have*

$$d_j^{(1)} > 0 \Rightarrow m_{P_j^{(1)}}(mC) = m$$

That is, if the exceptional curve $E_j^{(1)}$ appears as a component in F (with multiplicity $d_j^{(1)} > 0$) then C is smooth at the point $P_j^{(1)}$.

Proof. If the exceptional curve $E_j^{(1)}$ appears as a component in F, then $mE_j^{(1)}$ cannot appear as a component of the multiple fiber mE. Hence $m_{P_j^{(1)}}(mC) - m = 0$. □

Corollary II.3.1.22. *If C is singular at a base point $P_j^{(1)}$, then $m_{P_j^{(1)}}(B) = m$. Moreover, at the point $P_j^{(1)}$ the curve B consists of a single component (branch) with multiplicity m.*

Proof. It follows from Lemma II.3.1.21 that if C is singular at a base point $P_j^{(1)}$, then $E_j^{(1)}$ is not a component of F, hence $d_j^{(1)} = 0$, which further implies $m_{P_j^{(1)}}(B) = m$. The last statement is obvious, otherwise one would need to blow-up more than one point lying in $E_j^{(1)}$ in order to separate $\mathcal{P}$. □

Since we are working over a field of characteristic zero, the unique multiple fiber mE can only be of multiplicative type, i.e. of type I_n. If $n \leq 3$, then mE can be realized as the strict transform (under π) of the unique multiple cubic mC. But if $n > 3$, then, necessarily, C must be singular at a base point of $\mathcal{P}$. In other words,

Lemma II.3.1.23. *If Y contains a multiple fiber of type I_n with $4 \leq n \leq 9$, then C is singular at a base point of $\mathcal{P}$.*

Proof. If C is smooth at every base point of $\mathcal{P}$, then the corresponding multiple fiber on Y is given by $m\overline{C} + m \cdot \sum_{i,j}(m_{P_j^{(i)}}(C) - 1)E_j^{(i)} = m\overline{C}$, where $\overline{C}$ is the strict transform of C under π and $m_{P_j^{(i)}}(C) = 1$ is the multiplicity of $P_j^{(i)}$ on the strict transform of

C under the blow-up of $P_j^{(1)}, \ldots, P_j^{(i-1)}$. That is, the multiple fiber of Y is simply given by the strict transform of mC. But each of the fibers $I_n (4 \leq n \leq 9)$ have at least four components and hence the corresponding multiple fiber cannot be realized as strict transforms of a multiple cubic in the plane, a contradiction. $\square$

When C is singular at a base point of $\mathcal{P}$, it is also useful and interesting to understand how singular it can be.

Proposition II.3.1.24. *For any index m we have that $lct(\mathbb{P}^2, mC) = \dfrac{1}{m}$.*

Proof. If C is irreducible, then there is nothing to prove. Otherwise, we claim that C consists of either a conic and a line intersecting it transversally or three distinct lines in general position (i.e. not concurrent at a point).

Clearly C cannot be non-reduced so we must exclude the following three cases:

(a) a cusp

(b) a conic and a tangent line

(c) three concurrent lines

Because the unique multiple fiber of Y can only be of type $I_n, n \leq 9$, in any of the above cases the singular point of C must be a base point of the pencil $\mathcal{P}$. Moreover, since (c) can be obtained as soon as one blow-up (of the tangency point) is performed in a cubic as in (b) and, in turn, (b) can be obtained as soon as one blow-up (of the cusp) is performed in a cubic as in (a), it suffices to consider only case (c). But blowing-up the concurrency point yields a component with multiplicity $2m$, which is an absurd. Such component is not a multisection of degree m and it cannot be a component in the multiple fiber either. $\square$

Proposition II.3.1.25. *If F is of type IV^* or III^*, then C is singular at most one base point of $\mathcal{P}$.*

Proof. From the proof of Proposition II.3.1.24 we know that C is reduced and either C is irreducible or it consists of a conic and a line intersecting transversally or three lines in general position. Moreover, from the classification in [43] we also know that if F is of type IV^* (resp. III^*), then the multiple fiber mE can only be of type I_0, I_1, I_2 or I_3 (resp. I_0, I_1 or I_2). Now, if C were singular at more than one base point of $\mathcal{P}$, then C would necessarily consist of a conic and a line intersecting transversally and the two intersecting points would be base points of $\mathcal{P}$. But then we would need to blow-up each of those two points at least twice, which would yield at least two more components in the multiple fiber. That is, mE would be of type I_n with $n \geq 4$, a contradiction. $\square$

Remark II.3.1.26. *If F is of type II^*, then C must be smooth at every base point of $\mathcal{P}$, because E is of type I_0 or I_1* [43]*. In particular, E (hence C) is irreducible, π restricts to an isomorphism $E \simeq C$ and C cannot be singular at any base point of $\mathcal{P}$.*

II.3.1.1.2 The singularities of B and the log canonical threshold

We are now ready to study the singularities of the curve B in terms of the type of the (non-multiple) fiber F. We investigate the multiplicities of B at the base points of $\mathcal{P}$ and we compute bounds for the log canonical threshold of the pair $(\mathbb{P}^2, B)$ by establishing some relations between the log canonical thresholds of the pairs (Y, F) and $(\mathbb{P}^2, B)$.

We begin by proving the following Lemma:

Lemma II.3.1.27. *If $\mathcal{P}$ does not contain an infinitely near point as a base point (i.e $a_j = 1$ for all $j = 1, \ldots, k$), then $k = 9$ and $F = \overline{F} + \sum_{j=1}^{9}(m_{P_j^{(1)}}(B) - m)E_j^{(1)} = \overline{F}$.*

Proof. If $a_j = 1$ for all $j = 1, \ldots, k$, then it is clear that $k = 9$, since $a_1 + \ldots + a_k = 9$. Moreover, $0 = d_j^{(a_j)} = d_j^{(1)} = m_{P_j^{(1)}}(B) - m$ for all $j = 1, \ldots, 9$. □

Corollary II.3.1.28. *Let S_F denote the sum of all the multiplicities of the components of a fiber F and let n_F denote the number of its components. If either $S_F > 3m$ or $n_F > 3m$, then $\mathcal{P}$ must contain an infinitely near point as a base point. In particular, there exists some $1 \leq j \leq k$ so that $a_j > 1$ and $d_j^{(1)} \geq 1$.*

Proof. If $\mathcal{P}$ does not contain an infinitely near point as a base point, then Lemma II.3.1.27 tells us F is the strict transform of a member of $\mathcal{P}$, which implies both $S_F \leq 3m$ and $n_F \leq 3m$. □

Corollary II.3.1.29. *Using the same notations as in Corollary II.3.1.28, if a fiber F is such that $S_F > 3m$ or $n_F > 3m$, then there exists a base point $P_j^{(1)}$ in $\mathcal{P}$ such* *that* $m_{P_j^{(1)}} \geq m+1$.

Proof. By Corollary II.3.1.28 there exists some j so that $d_j^{(1)} \geq 1$ and the result follows from the equality $d_j^{(1)} = m_{P_j^{(1)}}(B) - m$. □

We also prove the following:

Lemma II.3.1.30. *If M_F denotes the largest multiplicity of a component of F, then every base point $P_j^{(1)}$ of $\mathcal{P}$ is such that $m_{P_j^{(1)}}(B) \leq \min\{M_F + m, 3m\}$.*

Proof. If follows from the fact that B has degree $3m$ and $M_F \geq d_j^{(1)} = m_{P_j^{(1)}}(B) - m$. □

Corollary II.3.1.31. *If F is non-reduced and $m \leq M_F$, then every base point $P_j^{(1)}$ of $\mathcal{P}$ is such that $m_{P_j^{(1)}} \leq 2M_F$.*

Lemma II.3.1.32. *If F is of type II, III or IV, then $F = \overline{F}$*

Proof. If F is of type II, III or IV we are claiming F cannot contain any exceptional curves $E_j^{(i)}$. This is clear when F is of type II. If F is of type III, then F contains exactly two rational components which are tangent at a single point Q with multiplicity two. If one of these components is equal to $E_j^{(1)}$ for some j, then the cubic C must intersect B at a base point $P_j^{(}1)$ with multiplicity $m+1$ in B and, which after one blow-up, becomes the tangency point between (the strict transform of) B and $E_j^{(1)}$. But after the first blow-up (the strict transform of) C would also go through the tangency point, hence we would have $Q = P_j^{(2)}$ and blowing-up $P_j^{(2)}$ to separate the pencil would not yield the desired type of fiber.

The argument is analogous for F of type IV. $\square$

Proposition II.3.1.33. *If F is reduced, then B is reduced and,*

$$\frac{1}{m+1} < lct(\mathbb{P}^2, B) = \min\left\{lct_{P_j^{(1)}}(\mathbb{P}^2, B), lct(Y, F)\right\} \le lct(Y, F) \le lct(Y, \overline{F})$$

Proof. We first show the equality. We have that $lct(\mathbb{P}^2, B) = \min_P\{lct_P(\mathbb{P}^2, B)\}$, where P runs over the singular points of B. But any singular point of B is either a base point of $\mathcal{P}$ of it is not a base point and hence it must satisfy $lct_P(\mathbb{P}^2, B) = lct_P(Y, \overline{F})$. Moreover, $lct_P(Y, \overline{F}) = lct(Y, F)$, because either F is of type II, III or IV and F contains a unique singular point, namely (the strict transform of) P; or F is of type $I_n, 1 \le n \le 9$ and every singular point of F is an ordinary node and we have that $lct_P(Y, \overline{F}) = lct(Y, F) = 1$.

Now, because F is reduced we have

$$\frac{1}{m+1} \le \frac{1}{2} < lct(Y, F)$$

On the other hand, it follows from Lemma II.3.1.32 that for any singular point of B which is a base point of $\mathcal{P}$, say P_j, we have

$$lct_{P_j^{(1)}}(\mathbb{P}^2, B) = \frac{1 + b^{(1)}}{c_j^{(1)}} = \frac{2}{c_j^{(1)}} = \frac{2}{m} > \frac{1}{m}$$

Finally, it is clear that (see e.g. [35, Theorem 8.20]) $lct(Y, F) \le lct(Y, \overline{F})$ because

$$F = \overline{F} + \sum_{i,j} d_j^{(i)} E_j^{(i)}$$

$\square$

Proposition II.3.1.34. *If $m > 1$ and F is reduced, then $lct(\mathbb{P}^2, B) > \dfrac{1}{m}$.*

Proof. It follows from the proof of Proposition II.3.1.33 by observing that for $m > 1$ we have $lct(Y, F) > \dfrac{1}{2} \geq \dfrac{1}{m}$. $\square$

Proposition II.3.1.35. *If F is non-reduced and $m \leq M_F$, where M_F denotes the largest multiplicity of a component of F, then*

$$lct(Y, F) \leq lct(\mathbb{P}^2, B) \leq lct(Y, \overline{F})$$

Proof. If F is non-reduced, then $\pi : Y \to \mathbb{P}^2$ is a log resolution of the pair $(\mathbb{P}^2, B)$ (see Definition I.3.0.2) and it follows that

$$lct(\mathbb{P}^2, B) = \min_{i,j} \left\{ \frac{1 + b_j^{(i)}}{c_j^{(i)}}, \frac{1}{M_B} \right\} \leq \frac{1}{M_B} = lct(Y, \overline{F}) \tag{II.3.1.9}$$

where M_B denotes the largest multiplicity of a component of B.

If $lct(\mathbb{P}^2, B) = 1/M_B$ there is nothing to prove, since $M_B \leq M_F$ and we have $lct(Y, F) = 1/M_F$.

Thus, assume there exists some i and some j such that

$$lct(\mathbb{P}^2, B) = \frac{1 + b_j^{(i)}}{c_j^{(i)}} < \frac{1}{M_F} \leq \frac{1}{M_B}$$

If $i = 1$, then

$$\frac{1 + b_j^{(i)}}{c_j^{(i)}} = \frac{2}{m_{P_j^{(1)}}} < \frac{1}{M_F} \iff m_{P_j^{(1)}} > 2M_F$$

which contradicts Corollary II.3.1.31.

Similarly, if $i = 2$, then

$$\frac{1+b_j^{(i)}}{c_j^{(i)}} = \frac{3}{m_{P_j^{(1)}} + m_{P_j^{(2)}}} < \frac{1}{M_F} \iff m_{P_j^{(1)}} + m_{P_j^{(2)}} > 3M_F$$

but $m_{P_j^{(1)}} + m_{P_j^{(2)}} = d_j^{(2)} + 2m \leq M_F + 2m \leq 3M_F$

Otherwise, using Equation II.3.1.5, we can write $c_j^{(i)} = d_j^{(i)} + i \cdot m$. Then,

$$\begin{aligned} \frac{1+b_j^{(i)}}{c_j^{(i)}} = \frac{1+b_j^{(i)}}{d_j^{(i)} + i \cdot m} < \frac{1}{M_F} \quad &\iff \quad M_F(1+b_j^{(i)}) < d_j^{(i)} + i \cdot m \\ &\iff \quad M_F(1+i) < d_j^{(i)} + i \cdot m \end{aligned}$$

which is a contradiction because $M_F \geq d_j^{(i)}$ and $M_F \geq m$. □

Remark II.3.1.36. *Note that Equation (II.3.1.9) in the proof of Proposition II.3.1.35 holds for any index m. In particular, if F is of type I_n^*, II^*, III^* or IV^*, then we also have that (see e.g. [11]) $\frac{1}{m_{P_{jmax}^{(1)}}} \leq lct(\mathbb{P}^2, B)$, where $m_{P_{jmax}^{(1)}} \doteq \max_j m_{P_j^{(1)}}(B)$.*

Then Propositions II.3.1.16 and II.3.1.17 allow us to further prove:

Proposition II.3.1.37. *For any index m we have $lct(Y, \overline{F}) \leq 2lct(Y, F)$.*

Proof. By contradiction, assume $1/M_B > 2lct(Y, F)$. If F does not contain a component with multiplicity ≥ 3, then $2lct(Y, F) \geq 1$ and we conclude $M_B < 1$, a contradiction. If F is of type III^* or IV^*, then B must be reduced (i.e., $M_B = 1$) and if F is of type II^*, then we conclude $M_B < 3$, contradicting Propositions II.3.1.16 and II.3.1.17. □

Remark II.3.1.38. *Note that when F is of type II^*, III^* or IV^*, then the inequality $1/M_B \leq 2lct(Y, F)$ implies Propositions II.3.1.16 and II.3.1.17.*

In particular, combining Propositions II.3.1.33, II.3.1.35 and II.3.1.37 we obtain:

Corollary II.3.1.39. *For any index m we have $lct(\mathbb{P}^2, B) \leq 2lct(Y, F)$.*

II.3.2 The stability criteria

We are finally ready to complete characterize the (semi)stability of Halphen pencils of index two under the action of $SL(3)$ (as points in $Gr(2, 28)$) .

Recall that any Halphen pencil of index two (Definition II.3.1.4) contains exactly one multiple cubic $2C$ (of multiplicity two), which corresponds to the unique multiple fiber in the associated rational elliptic surface. Thus any Halphen pencil $\mathcal{P}$ of index two can be written in the following form: $\lambda(B) + \mu(2C) = 0$, where the curve B corresponds to some (non-multiple) fiber of Y that we denote by F.

With these notations in mind we will first establish necessary conditions for nonstability and unstability of a Halphen pencil of index two:

Theorem II.3.2.1. *If $\mathcal{P}$ is not stable, then Y contains a non-reduced fiber*[3]*.*

Proof. Since $lct(\mathbb{P}^2, 2C) = \dfrac{1}{2}$ Proposition II.3.1.24 (= [54, Proposition 4.9]), we conclude from Theorem II.2.3.5 (= [55, Theorem 1.1]), with $\alpha = \dfrac{1}{2}$, that if the pencil $\mathcal{P}$ is not stable, then $\mathcal{P}$ contains a curve B such that $lct(\mathbb{P}^2, B) \leq \dfrac{1}{2}$. By Proposition II.3.1.34 (= [54, Proposition 4.15]) this implies the corresponding rational elliptic surface $Y \to \mathbb{P}^1$ contains a non-reduced fiber F. $\square$

[3] i.e. a fiber of type I_n^*, II^*, III^* or IV^*

Remark II.3.2.2. *A completely analogous argument in fact shows the statement of Theorem II.3.2.1 is true for Halphen pencils of any index.*

Theorem II.3.2.3. *If $\mathcal{P}$ is unstable, then Y contains a fiber of type II^*, III^* or IV^*.*

Proof. The proof is very similar to the proof of Theorem II.3.2.1. Since we know $lct(\mathbb{P}^2, 2C) = \dfrac{1}{2}$ Proposition II.3.1.24 (= [54, Proposition 4.9]), we conclude from Theorem II.2.3.5 (= [55, Theorem 1.1]), by taking $\alpha = \dfrac{1}{2}$, that if the pencil $\mathcal{P}$ is unstable, then $\mathcal{P}$ contains a curve B such that $lct(\mathbb{P}^2, B) < \dfrac{1}{2}$. Thus, Propositions II.3.1.34 and II.3.1.35 (= [54, Propositions 4.15 and 4.16]) imply Y contains a a fiber of type II^*, III^* or IV^*. □

The next step is to obtain sufficient conditions. When C is smooth and B is semistable we prove:

Proposition II.3.2.4. *If C is smooth and all curves in $\mathcal{P}$ are stable except (possibly) for one curve that is semistable, then $\mathcal{P}$ is stable.*

Proof. It follows from Theorem II.2.2.12 (= [55, Theorem 1.5]) and the fact that $2C$ is stable [52]. □

Corollary II.3.2.5. *If C is smooth, F is of type II^*, III^* or IV^* and $B \doteq \pi(F)$ is semistable, then $\mathcal{P}$ is stable.*

Proof. From the classification in [47] we know that any other fiber of Y is reduced. By Propositions II.3.1.33 and II.3.1.34 (= [54, Propositions 4.14 and 4.15]) we also know that all other curves in $\mathcal{P}$ are reduced and have log canonical threshold greater than 1/2. As observed in [23] and [27], this implies all the curves in $\mathcal{P}$ are stable except for one curve that is semistable. $\square$

Corollary II.3.2.6. *If C is smooth and Y contains exactly one non-reduced fiber F of type $I_n^*, n \leq 4$, then $\mathcal{P}$ is stable.*

Proof. Again, from the classification in [47] we know that any other fiber of Y is reduced. Since the curve B is such that $lct(\mathbb{P}^2, B) \geq 1/2$, hence it is semistable [23, 27], we can argue as in the proof of Corollary II.3.2.5 to conclude all the curves in $\mathcal{P}$ are stable except (possibly) for one curve that is semistable. $\square$

Theorem II.3.2.7. *If Y contains two fibers of type I_0^*, then $\mathcal{P}$ is strictly semistable if and only if there exists a one-parameter subgroup λ (and coordinates in $\mathbb{P}^2$) such that the two curves corresponding to the fibers of type I_0^* are both non-stable with respect to this λ.*

Proof. By Proposition II.3.1.35 (= [54, Proposition 4.16]), if F is a fiber of type I_0^*, then the corresponding plane curve B is such that $lct(\mathbb{P}^2, B) \geq \frac{1}{2}$, hence it is semistable [23, 27]. The result then follows from Theorem II.2.2.13 (= [55, Theorem 1.6]). Note that from the topological Euler characteristic of Y we know C has to be smooth, hence stable [52]. $\square$

And when C is singular we prove:

Theorem II.3.2.8. *If C is singular and Y contains exactly one fiber F of type $I_n^*, n \leq 4$, then $\mathcal{P}$ is strictly semistable if and only if there exists a one-parameter* *subgroup λ (and coordinates in $\mathbb{P}^2$) such that $2C$ and $B = \pi(F)$ are both non-stable with respect to this λ.*

Proof. Since both $2C$ and B are semistable and all other curves in $\mathcal{P}$ are stable, the result follows from Theorem II.2.2.13 (= [55, Theorem 1.6]). □

Theorem II.3.2.9. *If C is singular, Y contains a fiber F of type II^*, III^* and IV^* and the curve $B = \pi(F)$ is semistable, then $\mathcal{P}$ is strictly semistable if and only if there exists a one-parameter subgroup λ (and coordinates in $\mathbb{P}^2$) such that $2C$ and B are both non-stable with respect to this λ.*

Proof. Again, the result follows from Theorem II.2.2.13 (= [55, Theorem 1.6]) because both $2C$ and B are semistable and all other curves in $\mathcal{P}$ are stable. □

Finally, in order to complete our description, we need to study the stability of $\mathcal{P}$ when F is a fiber of type II^*, III^* or IV^*.

II.3.2.1 The stability of $\mathcal{P}$ when F is of type II^*

When F of type II^*, then Theorem A.1.4 (= [54, Theorem 5.15]) tells us B can only be realized by one of the following plane curves:

(i) a triple conic

(ii) a nodal cubic and an inflection line, with the line taken with multiplicity three

(iii) two triples lines

(iv) a conic and a tangent line, with the line taken with multiplicity four

(v) a line with multiplicity five and another line

If B is a triple conic, then B is strictly semistable [52]. In this case, if C is smooth, then $\mathcal{P}$ is stable (Corollary II.3.2.5) and if C is singular, then $\mathcal{P}$ is strictly semistable if and only if there exists a one-parameter subgroup λ (and coordinates in $\mathbb{P}^2$) such that $2C$ and B are both non-stable with respect to this λ (Theorem II.3.2.9).

When B is one of the curves in $(ii), (iii), (iv)$ or (v) then we can use the explicit constructions obtained in [54] and described in Appendix A to conclude $\mathcal{P}$ is unstable.

More precisely, we prove Propositions II.3.2.11 through II.3.2.13 below.

Proposition II.3.2.10. *If Y contains a fiber of type II^* and $\mathcal{P}$ contains a curve consisting of two triple lines, then $\mathcal{P}$ is unstable.*

Proof. Let $\mathcal{P}$ and Y be as above. One can show that one of the lines is an inflection line of C and the other line must be tangent to the cubic with multiplicity two (Example A.2.26).

In particular, we can find coordinates in $\mathbb{P}^2$ so that B is given by $x^3y^3 = 0$ and C is given by $z^2x - y(y-x)(y-\alpha\cdot x) = 0$, where $\alpha \in \mathbb{C}\backslash\{0,1\}$. Then the Plücker coordinates of $\mathcal{P}$ with respect to these coordinates satisfy the conditions in Case (1) of Theorem II.2.5.6 and we conclude $\mathcal{P}$ is unstable. Alternatively, we can easily check the equations for B and $2C$ belong to Case 5 of Theorem II.2.5.12. $\square$

Proposition II.3.2.11. *If Y contains a fiber of type II^* and $\mathcal{P}$ contains a curve consisting of a triple line and a nodal cubic, then $\mathcal{P}$ is unstable.*

Proof. Let $\mathcal{P}$ and Y be as above. One can show that the line is an inflection line of both the nodal cubic and C, which is smooth (Example A.2.27).

In particular, we can find coordinates in $\mathbb{P}^2$ so that the curve B has equation $x^3(xz^2 - y^2(y+x)) = 0$ and C is given by $x^2y + xz^2 - y^3 - xy^2 = 0$. Then the Plücker coordinates of $\mathcal{P}$ with respect to these coordinates satisfy the conditions in Case (1) of Theorem II.2.5.6 and we conclude $\mathcal{P}$ is unstable. Alternatively, we can easily check the equations for B and $2C$ belong to Case 4 of Theorem II.2.5.12.

□

Proposition II.3.2.12. *If Y contains a fiber of type II^* and $\mathcal{P}$ contains a curve consisting of a conic and a tangent line, with the line taken with multiplicity four, then $\mathcal{P}$ is unstable.*

Proof. Let $\mathcal{P}$ and Y be as above. One can show that C must be tangent to the conic (resp. the line) at the point $Q \cap L$ with multiplicity six (resp. two) as in Example A.2.28.

In particular, we can find coordinates in $\mathbb{P}^2$ so that B is given by the zeros of the polynomial $x^4(y^2 + xz)$ and C is given by $f = \sum f_{ij}x^iy^jz^{6-i-j} = 0$, with $f_{00} = f_{01} = f_{02} = 0$. Thus, the Plücker coordinates of $\mathcal{P}$ with respect to these coordinates satisfy the conditions in Case (1) of Theorem II.2.5.6 and we conclude $\mathcal{P}$ is unstable. Alternatively, we can easily check the equations for B and $2C$ belong to Case 2 of Theorem II.2.5.12. □

Proposition II.3.2.13. *If Y contains a fiber of type II^* and $\mathcal{P}$ contains a curve* *consisting of a line with multiplicity five and another line, then $\mathcal{P}$ is unstable.*

Proof. Let $B \in \mathcal{P}$ be the curve consisting of a line with multiplicity five and another line. We can choose coordinates so that B is the curve $x^5(x-z)=0$ and C is the cubic $y^2z = x(x-z)(x-\alpha \cdot z)$ for some $\alpha \in \mathbb{C}\backslash\{0,1\}$ (Example A.2.29). Then the Plücker coordinates of $\mathcal{P}$ satisfy the vanishing conditions of Case (1) in Theorem II.2.5.6. Or, yet, we can easily check the equations for B and $2C$ belong to Case 1 of Theorem II.2.5.12. □

II.3.2.2 The stability of $\mathcal{P}$ when F is of type III^*

We now consider the case when F is of type III^*.

From Theorem A.1.5 (= [54, Theorem 5.16]) the curve B can only be realized by one of the following plane curves:

(i) a double line, a cubic and another line

(ii) a double conic and another conic (semistable)

(iii) a triple conic (semistable)

(iv) two triple lines

(v) a triple line, a double line and another line

(vi) a triple line, a conic and a line

(vii) a triple line and a cubic

(viii) a conic and a line, with the line taken with multiplicity four

(ix) a line with multiplicity four and two other lines

If B is semistable there are two possibilities: either C is smooth, in which case $\mathcal{P}$ is stable (Corollary II.3.2.5); or C is singular and then $\mathcal{P}$ is strictly semistable if and only if there exists a one-parameter subgroup λ (and coordinates in $\mathbb{P}^2$) such that $2C$ and B are both non-stable with respect to this λ (Theorem II.3.2.9).

When B is unstable we can use the explicit constructions obtained in [54] to conclude $\mathcal{P}$ is strictly semistable.

Proposition II.3.2.14. *If Y contains a fiber F of type III^* and $B \doteq \pi(F)$ consists of a triple line, a double line and another line in general position, then $\mathcal{P}$ is not stable.*

Proof. Let $\mathcal{P}$ and Y be as above. One can find coordinates in $\mathbb{P}^2$ as in Example A.2.19 so that the Plücker coordinates of $\mathcal{P}$ with respect to these coordinates satisfy the conditions in Case (3) of Theorem II.2.5.7 and we conclude $\mathcal{P}$ is not stable. Alternatively, we can also apply Theorem II.2.5.10. $\square$

Lemma II.3.2.15. *If a Halphen pencil $\mathcal{P}$ of index two contains a curve B and a base point P such that $mult_P(B) = 6$, then $\mathcal{P}$ is not stable.*

Proof. Since $\mathrm{mult}_P(2C) \geq 2$, the result follows from Theorem II.2.4.1 (=[55, Theorem 1.3]). $\square$

Proposition II.3.2.16. *If Y contains a fiber F of type III^* and $B \doteq \pi(F)$ consists of a triple line, a double line and another line concurrent at a base point, then $\mathcal{P}$ is not stable.*

Proof. Let $\mathcal{P}, Y$ and B be as above. Then $\mathcal{P}$ contains a base point P (the point where the 3 lines meet) such that $\mathrm{mult}_P(B) = 6$, and the result follows from Lemma II.3.2.15. □

Proposition II.3.2.17. *If Y contains a fiber F of type III^* and $B \doteq \pi(F)$ consists of a double line, a nodal cubic and another line, then $\mathcal{P}$ is not stable.*

Proof. Let $\mathcal{P}$ and Y be as above. One can find coordinates in $\mathbb{P}^2$ as in Example A.2.15 so that the Plücker coordinates of $\mathcal{P}$ with respect to these coordinates satisfy the conditions in Case (3) of Theorem II.2.5.7 and we conclude $\mathcal{P}$ is not stable. □

Proposition II.3.2.18. *If Y contains a fiber F of type III^* and $B \doteq \pi(F)$ contains a line with multiplicity four, then $\mathcal{P}$ is not stable.*

Proof. If B contains a line with multiplicity four, then we can find coordinates in $\mathbb{P}^2$ and generators of $\mathcal{P}$ which are given by equations as in Case 1 of Theorem II.2.5.8. □

Proposition II.3.2.19. *If Y contains a fiber F of type III^* and $B \doteq \pi(F)$ consists of a triple line and a nodal cubic, then $\mathcal{P}$ is not stable.*

Proof. We can find coordinates in $\mathbb{P}^2$ as in Example A.2.22 so that the Plücker coordinates of $\mathcal{P}$ with respect to these coordinates satisfy the conditions in Case (4) of Theorem II.2.5.7 and we conclude $\mathcal{P}$ is not stable. Alternatively, we can also apply Theorem II.2.5.10. $\square$

Proposition II.3.2.20. *If Y contains a fiber F of type III^* and $B \doteq \pi(F)$ consists of a triple line, a conic and another line, then $\mathcal{P}$ is not stable.*

Proof. Let $\mathcal{P}$ and Y be as above. We can find coordinates in $\mathbb{P}^2$ as in Example A.2.21 so that the Plücker coordinates of $\mathcal{P}$ with respect to these coordinates satisfy the conditions in Case (3) of Theorem II.2.5.7 and we conclude $\mathcal{P}$ is not stable. Alternatively, we can also apply Theorem II.2.5.10. $\square$

Proposition II.3.2.21. *If Y contains a fiber F of type III^* and $B \doteq \pi(F)$ consists of two triple lines, then $\mathcal{P}$ is not stable.*

Proof. It follows from Lemma II.3.2.15. $\square$

Combining Propositions II.3.2.14 through II.3.2.21 and Theorem A.1.5 we obtain:

Theorem II.3.2.22. *If Y contains a fiber F of type III^* and $B \doteq \pi(F)$ is unstable, then $\mathcal{P}$ is not stable.*

Remark II.3.2.23. *Note that when F is of type III^* and $B \doteq \pi(F)$ is semistable we can refer to Corollary II.3.2.5 and Theorem II.3.2.9.*

So the remaining question is: Can $\mathcal{P}$ be unstable? We will show that the answer to this questions is no.

Lemma II.3.2.24. *Let $\mathcal{P}$ be a Halphen pencil of index two containing a curve B such that $B = 4L + Q$, where L is a line and Q is a conic (possibly reducible). Letting $2C$ denote the unique multiple cubic in $\mathcal{P}$ we have that if $\mathcal{P}$ is unstable, then either*

(i) L is an inflection line of C or

(ii) L is tangent to C at a point where L and Q also intersect

Proof. It follows from Theorem II.2.5.13. □

Proposition II.3.2.25. *If Y contains a fiber F of type III^* and $B \doteq \pi(F)$ contains a line with multiplicity four, then $\mathcal{P}$ is semistable.*

Proof. If $\mathcal{P}$ were unstable, then $\mathcal{P}$ (and B) would be as in (i) or (ii) in Lemma II.3.2.24. In Appendix A we show that this is not case for a fiber of type III^*. □

Lemma II.3.2.26. *Let $\mathcal{P}$ be a Halphen pencil of index two containing a curve B such that $B = 3L + C'$, where L is a line and C' is a cubic (possibly reducible). Letting $2C$ denote the unique multiple cubic in $\mathcal{P}$ we have that if $\mathcal{P}$ is unstable, then either*

1. *L is an inflection line of C at a point where the intersection multiplicity of L and C' is ≥ 2 or*
2. *L is tangent to C at a point where the intersection multiplicity of L and C' is three.*

Proof. It follows from Theorem II.2.5.14. □

Proposition II.3.2.27. *If Y contains a fiber F of type III^* and $B \doteq \pi(F)$ contains a triple line, then $\mathcal{P}$ is semistable.*

Proof. If $\mathcal{P}$ were unstable, then $\mathcal{P}$ (and B) would be as in (i) or (ii) in Lemma II.3.2.26. In Appendix A we show that this is not case for a fiber of type III^*. □

Proposition II.3.2.28. *If Y contains a fiber F of type III^* and $B \doteq \pi(F)$ consists of a double line, a cubic and another line, then $\mathcal{P}$ is semistable.*

Proof. It follows from Theorem II.2.5.15. □

II.3.2.3 The stability of $\mathcal{P}$ when F is of type IV^*

Finally, we describe the stability of $\mathcal{P}$ when F is of type IV^*. We will show that either $\mathcal{P}$ is stable or C is singular and B is semistable, in which case we can refer to Theorem II.3.2.9.

We start with the following Lemma:

Lemma II.3.2.29. *Let $\mathcal{P}$ be a Halphen pencil of index two containing a curve B such that $B = 3L + C'$, where L is a line and C' is a cubic (possibly reducible). Letting $2C$ denote the unique multiple cubic in $\mathcal{P}$ we have that if $\mathcal{P}$ is not stable, then either*

(i) L is an inflection line of C or

(ii) L is tangent to C at a point where L and C' also intersect or

(iii) there is a base point where L and C intersect and where the intersection multiplicity of L and C' is 3

Proof. It follows from Theorem II.2.5.10. $\square$

In particular, we conclude:

Proposition II.3.2.30. *If Y contains a fiber F of type IV^* and $B \doteq \pi(F)$ contains a triple line, then $\mathcal{P}$ is stable.*

Proof. If $\mathcal{P}$ were not stable, then $\mathcal{P}$ (and B) would be as in (i),(ii) or (iii) in Lemma II.3.2.29. In Appendix A we show that this is not case for a fiber of type IV^*. $\square$

We also prove:

Lemma II.3.2.31. *Let $\mathcal{P}$ be a Halphen pencil of index two containing a curve B such that $B = 2L + Q$, where L is a line and Q is a quartic (possibly reducible). Letting $2C$ denote the unique multiple cubic in $\mathcal{P}$ we have that if $\mathcal{P}$ is not stable, then the intersection multiplicity of L and Q at some base point is 4.*

Proof. It follows from Theorem II.2.5.11. $\square$

Lastly,

Theorem II.3.2.32. *If Y contains a fiber of type IV^* and $\mathcal{P}$ is not stable, then C is singular and B is semistable.*

Proof. If $\mathcal{P}$ is not stable, then it follows from Corollary II.3.2.5 that either C is singular or B is unstable. Now, the results from Appendix A and [52, Section 2] together with Proposition II.3.2.30 and Lemma II.3.2.31 imply B cannot be unstable. Thus, C is singular and B is semistable. $\square$

Note that the results above indeed give a complete description of the stability when F is of type IV^* because of Theorem A.1.6 (=[54, Theorem 5.17]). We know that when F is of type IV^*, then B consists of one of the following curves:

(i) a double conic and a conic (semistable)

(ii) a double line, a conic and two lines

(iii) a double line, a cubic and a line

(iv) a double line and two conics

(v) two double lines and two lines

(vi) two double lines and a conic

(vii) a double conic and two lines (semistable)

(viii) a triple conic (semistable)

(ix) a triple line, a conic and a line

(x) a triple line, a double line and another line

(xi) a triple line and three lines

(xii) a triple line and a cubic

Appendix A

Constructions of Halphen pencils of index two

It is well known that rational elliptic surfaces admitting a global section can be realized from a pencil of cubic curves in the plane (by blowing-up their nine base points) and explicit examples having a Mordell-Weil group with some particular rank have been considered in [16, Theorem 5.6.2], [20],[46] and [50]. However, there are not many explicit constructions in the literature for those rational elliptic surfaces that do not admit a global section. In [54], for each of the types of singular fibers that occur (see Proposition II.3.1.13) we constructed at least one explicit example of a rational elliptic surface $f : Y \to \mathbb{P}^1$ of index two having that type of singular fiber. In fact, for some types of singular fibers we constructed all possible examples.

The goal of this appendix is to present the examples we constructed in [54] for fibers of type II^*, III^* and IV^* since these constructions are particularly useful for obtaining the stability criteria from Chapter II.3, Section II.3.2.

Note that in view of Proposition II.3.1.9, these are obtained by explicitly constructing the corresponding Halphen pencils $\mathcal{P}$.

A.1 An algorithm

Adopting the same notations as in Section II.3.1.1, we first summarize what our strategy was for constructing the examples. Given F we know the number of its components n_F. Assuming we also know the number n_B of components of B we can compute k (the number of base points [1] in B) from Equation II.3.1.3 and Lemma II.3.1.19.

There are exactly $k - n_{E\setminus C}$ disjoint chains of rational curves in F as in Figure II.3.1, where $n_{E\setminus C}$ denotes the difference between the number of components of E and the number of components of C. Moreover, together with the strict transform of B under π these are all the components of F. Thus, analyzing how the dual graph of F must look like we can decide whether the components coming from B and these disjoint chains could possibly yield the given fiber.

The desired configuration of rational curves imposes restrictions on how the curves B and C can intersect and how the components of B must intersect. Since B and C can only intersect at base points of $\mathcal{P}$ we can use Equation (II.3.1.8). It also imposes

[1] not counting infinitely near points

restrictions on the multiplicities $d_j^{(1)}$ of the components $E_j^{(1)}$ appearing in F. Recall we have the following equality: $d_j^{(1)} = m_{P_j^{(1)}}(B) - 2$ (Equation (II.3.1.5)). In particular, we know what $m_{P_j^{(1)}}(B)$, the multiplicity of B at the base point $P_j^{(1)}$, must be.

In addition, every time we consider the dual graph of F we can color the components coming from B in blue and in black we indicate the missing components as in Lemma II.3.1.15. Then the possible configurations are those where the components in black are arranged in exactly $k - n_{E\setminus C}$ disjoint chains as in Figure II.3.1. In particular, every black node can only be connected to at most two other black nodes (Lemma II.3.1.15).

These considerations give us an algorithm to decide whether a sextic B can or cannot yield the desired type of fiber allowing us to prove Propositions A.1.1, A.1.2 and A.1.3 below, and to also construct all possible examples yielding a fiber of type II^*, III^* or IV^*.

We prove:

Proposition A.1.1 ([54, Proposition 5.1])**.** *If F is of type II^*, then B does not consist of any of the following curves:*

(i) a line with multiplicity 6

(ii) a line with multiplicity four and a double line

(iii) a triple line, a double line and another line

Proposition A.1.2 ([54, Propositions 5.2–5.9]). *If F is of type III^*, then B does not consist of any of the following curves:*

(i) *double line and a (rational) quartic*

(ii) *a double line and two conics*

(iii) *a double conic and a double line*

(iv) *a double conic and two lines*

(v) *two double lines and a conic*

(vi) *three double lines*

(vii) *two double lines and two other lines*

(viii) *a line with multiplicity four and a double line*

Proposition A.1.3 ([54, Propositions 5.12–5.14]). *If F is of type IV^*, then B does not consist of any of the following curves:*

(i) *double line and a rational quartic*

(ii) *three double lines*

(iii) *a double conic and a double line*

In particular, we obtain the following characterization for the curve B whenever F is of type II^*, III^* or IV^*:

Theorem A.1.4 ([54, Theorem 5.15]). *If F is of type II^*, then the sextic B consists of one of the following (non-reduced) curves:*

(i) a triple conic (Example A.2.25)

(ii) a nodal cubic and an inflection line, with the line taken with multiplicity three (Example A.2.27)

(iii) two triples lines (Example A.2.26)

(iv) a conic and a tangent line, with the line taken with multiplicity four (Example A.2.28)

(v) a line with multiplicity five and another line (Example A.2.29)

Theorem A.1.5 ([54, Theorem 5.16]). *If F is of type III^*, then B consists of one of the following curves:*

(i) a double line, a cubic and another line (Example A.2.15)

(ii) a double conic and another conic (Example A.2.16)

(iii) a triple conic (Example A.2.17)

(iv) two triple lines (Example A.2.18)

(v) a triple line, a double line and another line (Examples A.2.19 and A.2.20)

(vi) a triple line, a conic and a line (Example A.2.21)

(vii) *a triple line and a cubic (Example A.2.22)*

(viii) *a conic and a line, with the line taken with multiplicity four (Example A.2.23)*

(ix) *a line with multiplicity four and two other lines (Example A.2.24)*

Theorem A.1.6 ([54, Theorem 5.17]). *If F is of type IV^*, then B consists of one of the following curves:*

(i) *a double conic and a conic (Example A.2.3)*

(ii) *a double line, a conic and two lines (Example A.2.4)*

(iii) *a double line, a cubic and a line (Example A.2.5)*

(iv) *a double line and two conics (Example A.2.6)*

(v) *two double lines and two lines (Example A.2.7)*

(vi) *two double lines and a conic (Example A.2.8)*

(vii) *a double conic and two lines (Example A.2.9)*

(viii) *a triple conic (Example A.2.10)*

(ix) *a triple line, a conic and a line (Example A.2.11)*

(x) *a triple line, a double line and another line (Example A.2.12)*

(xi) *a triple line and three lines (Example A.2.13)*

(xii) *a triple line and a cubic (Example A.2.14)*

A.2 The explicit constructions

A.2.1 Type IV^*

We now construct all possible examples of Halphen pencils of index two that yield a fiber of type IV^* in the corresponding rational elliptic surface (Theorem A.1.6).

Definition A.2.1. *Given a cubic C, a conic Q and a point $P \in C$, we say Q is an osculating conic of C at P if $I_P(Q, C) \geq 5$, where $I_P(Q, C)$ denotes the intersection multiplicity of Q and C at P.*

Definition A.2.2. *Given a cubic C, any point on it where a tangent conic intersects C with multiplicity six is called a sextactic point. If C is smooth, there are exactly 27 such points and if C is nodal, then there only 3 sextactic points (see e.g.* [9],[10]*).*

Example A.2.3 (A double conic and a conic [54, Example 7.34])**.** *Consider a smooth cubic C and let P_1 be a sextactic point. Let Q_1 be the corresponding osculating conic. Assume we can construct another conic Q_2 so that Q_2 is tangent to both Q_1 and C at P_1 with multiplicity three, Q_2 intersects C at other three points P_2, P_3, P_4. Then the fourth intersection point between the two conics is different than the P_i's. Letting $B = Q_1 + 2Q_2$ we have that the pencil generated by B and $2C$ is a Halphen pencil of index two and the corresponding rational elliptic surface has a fiber of type IV^*.*

For instance, let C be the cubic given by $xz^2 + y^2z + x^3 = 0$, then we can let $P_1 = (0 : 0 : 1)$ and we have that Q_1 is the conic $y^2 + xz = 0$. Choosing Q_2 to be the conic $xy + y^2 + xz = 0$ we get the desired pencil.

Example A.2.4 (A double line, a conic and two lines [54, Example 7.35])**.** *Let Q be a (smooth) conic and choose $P_1 \in Q$. Let T be the tangent line to Q at P_1. Let L_1 be a line through P_1, intersecting Q at a second point P_2. Choose two other points P_3 and P_4 in Q, let L_2 be the line joining them and let $\{P_5\} = L_1 \cap L_2$. Assume we can construct a cubic C through $P_1, \ldots, P_5$ which is tangent to Q (resp. T) with multiplicity* 3 *(resp.* 2.*). Then C intersects T at another point P_6.*

Letting $B = 2T + Q + L_1 + L_2$ we have that the pencil generated by B and $2C$ is a Halphen pencil of index two and the corresponding rational elliptic surface has a fiber of type IV^*.

For instance, we can choose coordinates so that Q is the conic $y^2 + xz = 0$ and we can choose $P_1 = (0 : 0 : 1)$. Then T is the line $x = 0$. Choosing L_1 to be the line $x + y = 0$ we have that $P_2 = (-1 : -1 : 1)$. Now, if we choose P_4 and P_5 so that L_2 is the line $x+y+z$, then $P_5 = (-1 : 1 : 0)$ and C is the cubic $x^3+y^3+2xyz+y^2z+xz^2 = 0$. Thus, P_6 is the point $(0 : 1 : -1)$.

Example A.2.5 (A double line, a cubic and another line [54, Example 7.36])**.** *Let D be a nodal cubic and denote its node by P_5. Let P_1 be a flex point of D and denote the corresponding inflection line by L. Let L' be a line that intersects D at three other points P_2, P_3 and P_4. Assume we can construct a cubic C through $P_1, \ldots, P_5$ so that C is tangent to D (resp. L) at P_1 with multiplicity* 4 *(resp.* 3*).*

For instance, let D be the nodal cubic $y^2z - x^2(x + z) = 0$. Then $P_5 = (0 : 0 : 1)$ and we can let $P_1 = (0 : 1 : 0)$ so that L is the line $z = 0$. Choosing L' to be the line

$x + y + z = 0$ *we have that* C *is the cubic* $xyz + xz^2 + y^2z - x^3 = 0$.

Letting $B = 2L + L' + D$ *we have that the pencil generated by* B *and* $2C$ *is a Halphen pencil of index two and the corresponding rational elliptic surface has a fiber of type* IV^*.

Example A.2.6 (A double line and two conics [54, Example 7.37])**.** *Let* C *be a smooth cubic. Let* P_2 *be a flex point. There exists a line* L *through* P_2 *which is tangent to* C *at another point* P_1. *Then* P_1 *is a sextactic (see Definition A.2.2) point of* C.

In fact, by [54, Lemma 7.25] *we have* $2P_1 \oplus P_2 = 0$ *and* $3P_2 = 0$, *hence* $3(2P_1 \oplus P_2) = 6P_1 = 0$, *where* $\oplus$ *denotes the group law with another flex point taken as the origin. Again, using* [54, Lemma 7.25] *we conclude there exists an osculating conic which is tangent to* C *with multiplicity at* P_1.

Concretely, we can choose coordinates in $\mathbb{P}^2$ *so that* C *is the cubic given by*

$$y^2z = x(x - z)(x - \alpha \cdot z) \quad \alpha \in \mathbb{C}\backslash\{0, 1\}$$

and C *has a flex point at* $P_2 = (0 : 1 : 0)$. *The line* $x = 0$ *is tangent to* C *at* $P_1 = (0 : 0 : 1)$ *and the flex* P_2 *is a point in that line.*

Now, let ε_2 *be a two torsion point of* C. *Using the same argument as in* [54, Example 7.26], *we can always find three points* P_3, P_4 *and* P_5 *in* C *so that* $P_3 \oplus P_4 \oplus P_5 = \varepsilon_2$. *In particular,* $2P_3 \oplus 2P_4 \oplus 2P_5 = 0$ *and we claim we must have*

$$3P_1 \oplus P_3 \oplus P_4 \oplus P_5 = 0 \tag{A.2.1}$$

and

$$P_1 \oplus 2P_2 \oplus P_3 \oplus P_4 \oplus P_5 = 0 \tag{A.2.2}$$

In fact, if one of these sums is non zero, then adding the two equations we obtain

$$0 \neq 4P_1 \oplus 2P_2 \oplus 2P_3 \oplus 2P_4 \oplus 2P_5 = 4P_1 \oplus 2P_2 = 0$$

a contradiction.

Applying [54, Lemma 7.25] *two Equations (A.2.1) and (A.2.2) we conclude there exist two conics Q and Q' so that: $P_1, P_3, P_4, P_5 \in Q$, the cubic C is tangent Q at P_1 with multiplicity three, $P_1, P_2, P_3, P_4, P_5 \in Q'$ and the cubic C is tangent Q at P_2 with multiplicity two. Note that, by construction, L is also tangent to Q at P_1.*

Letting $B = 2L + Q + Q'$ we have that the pencil generated by B and $2C$ is a Halphen pencil of index two and the corresponding rational elliptic has a fiber of type IV^*.

Example A.2.7 (Two double lines and two other lines [54, Example 7.38])**.** *Let Q be a smooth conic. And choose three distinct points on Q say P_1, P_2 and P_3. For each $i = 1, 2$ let T_i be the tangent line to Q at P_i. Let L_i be the line joining P_1 and P_i, for $i = 2, 3$. And let L be a line through $\{P_4\} = T_1 \cap T_2$ different than the T_i and such that $P_3 \notin L$. Then L intersects both L_2 and L_3 at two other points $P_5 \in L_2$ and $P_6 \in L_3$.*

Letting C be the cubic $Q + L$ and B be the sextic $T_1 + T_2 + 2L_2 + 2L_3$ we have that the pencil $\mathcal{P}$ generated by B and $2C$ is a Halphen pencil of index two which yields a fiber of type IV^* in the associated elliptic surface.

Example A.2.8 (Two double lines and a conic [54, Example 7.39])**.** *Let Q be a smooth conic. And choose three distinct points on Q say P_1, P_2 and P_3. For each*

$i = 1, 2, 3$ let L_i be the tangent line to Q at P_i. Let L (resp. R) be the lines joining P_1 and P_3 (resp. P_2 and P_3). And let $\{P_4\} = L \cap L_2$ and $\{P_5\} = R \cap L_1$.

Then the cubic $C = L_1 + L_2 + L_3$ is such that the intersection multiplicity of Q and C at P_i, for $i = 1, 2, 3$ is two and the pencil $\mathcal{P}$ generated by $B = Q + 2L + 2R$ and $2C$ is a Halphen pencil of index two which yields a fiber of type IV^* in the associated elliptic surface. In fact the Jacobian fibration of such surface is the surface X_{431} in Miranda and Persson's list [44].

Concretely, we can choose coordinates in $\mathbb{P}^2$ so that Q is given by $x^2 - yz = 0$, $P_1 = (0 : 0 : 1), P_2 = (0 : 1 : 0)$ and $P_3 = (1 : -1 : -1)$. Then L_1 is the line $y = 0$, L_2 is the line $z = 0$ and L_3 is the line $2x + y + z = 0$. And, therefore, L and R are the lines $x + y = 0$ and $x + z = 0$, respectively. Moreover, $P_4 = (1 : -1 : 0)$ and $P_5 = (1 : 0 : -1)$.

Example A.2.9 (A double conic and two lines [54, Example 7.40])**.** *Let C be a smooth cubic. Let L_1 be an inflection line of C at a point P_1 and choose a line L_2 through P_1 which is tangent to C at another point P_2. We can construct a conic Q through P_1 and P_2 so that Q is tangent to C at P_1 with multiplicity two and Q meets C transversally at P_2. Moreover, Q intersects C at other three points, say P_3, P_4 and P_5.*

Concretely, choose coordinates in $\mathbb{P}^2$ so that C is the cubic given by

$$y^2 z = x(x - z)(x - \alpha \cdot z) \quad \alpha \in \mathbb{C} \backslash \{0, 1\}$$

Then we can let L_1 be the line $z = 0$ and hence $P_1 = (0 : 1 : 0)$ and we can let L_2 be *either one of the lines $x = 0, x - z = 0$ or $x - \alpha \cdot z = 0$.*

If we choose L_2 as $x = 0$, then $P_2 = (0 : 0 : 1)$ and, similarly, if we take L_2 as $x - z = 0$ (resp. $x - \alpha \cdot z = 0$), then $P_2 = (1 : 0 : 1)$ (resp. $P_2 = (\alpha : 0 : 1)$).

Say we choose L_2 to be the line $x = 0$, then we can let Q be the conic $x^2 + yz = 0$.

Now, the pencil $\mathcal{P}$ generated by $B = 2Q + L_1 + L_2$ and $2C$ is a Halphen pencil of index two that yields a fiber of type IV^* in the corresponding rational elliptic surface.

Example A.2.10 (A triple conic [42, I.5.11],[54, Example 7.41])**.** *In this example we consider a rational elliptic surface of index two whose Jacobian is the surface X_{431} in Miranda and Persson's list* [44].

Let $Q \subset \mathbb{P}^2$ be a smooth conic and choose three distinct points P_1, P_2 and P_3 on Q. Let L_i be the line tangent to Q at P_i and consider the pencil generated by $B = 3Q$ and $2C$, where $C = L_1 + L_2 + L_3$.

Note that we need to blow-up each of the three points three times. That is, to construct the desired surface we blow-up $\mathbb{P}^2$ at

$$P_1^{(1)}, P_1^{(2)}, P_1^{(3)}, P_2^{(1)}, P_2^{(2)}, P_2^{(3)}, P_3^{(1)}, P_3^{(2)}, P_3^{(3)}$$

which produces three disjoint chains of (-2)-curves, each of length 2 and formed by exceptional divisors over the corresponding three points.

Example A.2.11 (A triple line, a conic and another line [54, Example 7.42])**.** *Choose two (distinct) lines L_1 and L_2 and a smooth conic Q in general position. Let $\{P_2\} = L_1 \cap L_2$, let $\{P_2, P_4\} = L_1 \cap Q$ and let $\{P_1, P_3\} = L_3 \cap Q$. We can find a cubic C so that $P_1, P_2, P_3, P_4, P_5 \in C$ and C is tangent to Q at P_3 with multiplicity three.*

Concretely, we can choose coordinates in $\mathbb{P}^2$ *so that* Q *is the conic* $x^2+yz+xz=0$ *and* L_1 *and* L_2 *are the lines* $x+2y+z=0$ *and* $x=0$, *respectively.*

Then $P_1=(0:1:0), P_2=(0:1:-2), P_3=(0:0:1), P_4=(1:0:-1)$ *and* $P_5=(1:-1:1)$ *and we have that* C *is the cubic given by*

$$xy(x+z)+(x^2+yz+xz)(2y+z)=0$$

Now, the pencil generated by $B=Q+L_1+3L_2$ *and* $2C$ *is a Halphen pencil of index two which yields a fiber of type* IV^* *in the associated elliptic surface.*

Example A.2.12 (A triple line, a double line and another line [54, Example 7.43])**.** *Let* C *be a smooth cubic and let* L_1 *be an inflection line of* C *at a point* P_1. *We can choose another line* L_2 *through* P_1 *which is tangent to* C *at another point* P_2. *Let* L_3 *be a third line which intersects* C *at three distinct points, say* P_3, P_4 *and* P_5, *all different than* P_1 *and* P_2. *Then the pencil* $\mathcal{P}$ *generated by* $B=L_1+3L_2+2L_3$ *and* $2C$ *is a Halphen pencil of index two and it yields a fiber of type* IV^* *in the corresponding elliptic surface.*

Concretely, we can choose coordinates in $\mathbb{P}^2$ *so that* C *is the cubic given by*

$$y^2z=x(x-z)(x-\alpha\cdot z) \qquad \alpha\in\mathbb{C}\backslash\{0,1\}$$

we can let L_1 *be the line* $z=0$ *(hence* $P_1=(0:1:0)$*) and we can choose* L_2 *to be either one of the lines* $x=0, x-z=0$ *or* $x-\alpha\cdot z=0$.

If we choose L_2 *as* $x=0$, *then* $P_2=(0:0:1)$ *and we can let* L_3 *be the line* $x+y+z=0$.

Example A.2.13 (A triple line and three more lines [54, Example 7.44]). *Consider four (distinct) lines L_1, L_2, L_3 and L_4 in general position. That is, such that the L_i determine six intersection points, say $P_1, \ldots, P_6$. Now, choose a cubic C through these six points so that C intersects each of the lines transversally, i.e. the L_i are not tangent lines to C.*

The pencil $\mathcal{P}$ generated by $B = L_1 + L_2 + L_3 + 3L_4$ and $2C$ is a Halphen pencil of index two and it yields a fiber of type IV^* in the corresponding rational elliptic surface.

Example A.2.14 (A triple line and a cubic [54, Example 7.45]). *Let $D : d = 0$ be a nodal cubic with node at a point P_4. Let $L_1 : l_1 = 0$ and $L_2 : l_2 = 0$ be two of its inflections lines at points P_1 and P_2 ($\neq P_4$), respectively. And let L_3 be a line through the node P_4 which does not contain the flex points P_1 and P_2. Then the cubic C given by $l_1 l_2 l_3 + d = 0$ is such that the intersection multiplicity of D and C at P_i for $i = 1, 2$ is $I_{P_i}(C, D) = I_{P_i}(l_i, d) = 3$ and, by construction, the node P_4 lies on it.*

Now let L be the line joining P_1 and P_2. Then L intersects D at a third (flex) point P_3 and we have that the pencil $\mathcal{P}$ generated by $B = D + 3L$ and $2C$ is a Halphen pencil of index two which yields a fiber of type IV^* in the corresponding elliptic surface.

Concretely, we can choose as D the nodal cubic given by $z^3 + y^3 + xyz = 0$ with a node at the point $P_4 = (1 : 0 : 0)$. We can let L_1 be the line $-x + 3y + 3z = 0$, hence $P_1 = (0 : -1 : 1)$. And we can let L_2 be the line $-\omega x + 3y + 3\omega^2 z = 0$, where $\omega^3 = 1$. Then $P_2 = (0 : -1 : \omega)$ and L is the line $x = 0$. Note that L intersects D at the third *flex of D, namely $P_3 = (0 : -1 : \omega^2)$. Moreover, we can take as L_3 the line $z = 0$.*

A.2.2 Type III^*

We now construct all possible examples of Halphen pencils of index two that yield a fiber of type III^* in the corresponding rational elliptic surface (Theorem A.1.5).

Example A.2.15 (A double line, a cubic and another line [54, Example 7.46])**.** *Let D be a nodal cubic and denote its node by P_1. Let P_2 be a flex point of D and denote the corresponding inflection line by L_1. Let L_2 be a line through P_2 so that L_2 intersects D at two other points, say P_3 and P_4. We can construct a cubic C through $P_1, \ldots, P_4$ so that C is tangent to D (resp. L_1) at P_2 with multiplicity five (resp. three).*

Concretely, let D be the nodal cubic given by $y^2z - x^2(x + z) = 0$. Then $P_1 = (0 : 0 : 1)$ and we can let L_1 be the line $z = 0$, hence $P_2 = (0 : 1 : 0)$. Thus we can take L_2 to be the line $x - z = 0$. And, further, we have that $P_4 = (1 : \sqrt{2} : 1)$ and $P_5 = (1 : -\sqrt{2} : 1)$. Choosing C to be the cubic given by $y^2z - x(x^2 + z^2) = 0$ we have that all the points $P_1, \ldots, P_4$ lie in C and, moreover, the intersection multiplicity of C and D (resp. L_1) at P_2 is five (resp. three).

Now, the pencil $\mathcal{P}$ generated by $B = D + 2L_1 + L_2$ and $2C$ is a Halphen pencil of index two which yields a fiber of type III^* in the corresponding rational elliptic surface.

Example A.2.16 (A double conic and another conic [54, Example 7.47])**.** *Let Q be a conic and choose a point $P_1 \in Q$. We can construct another conic Q' and a* *smooth cubic C so that Q is tangent to both C and Q' at P_1 with full multiplicity and, moreover, the intersection multiplicity of Q' and C at P_1 is four and Q' intersects C at two other points, say P_2 and P_3.*

Concretely, choose coordinates in $\mathbb{P}^2$ so that Q is the conic given by $x^2+yz=0$ and let P_1 be the point $(0:0:1)$. Then we can let Q' be the conic given by $x^2+yz+y^2=0$ and we can let C be the cubic given by $y^3+z(x^2+yz)=0$. Thus, $P_2=(\alpha:1:1)$ and $P_3=(-\alpha:1:1)$, where $\alpha^2+2=0$

Now, the pencil $\mathcal{P}$ generated by $B=2Q'+Q$ and $2C$ is a Halphen pencil of index two such that the corresponding elliptic surface has a fiber of type III^*.

Example A.2.17 (A triple conic [54, Example 7.48])**.** *In this new example we construct a rational elliptic surface whose Jacobian is the surface X_{321} in Miranda and Persson's list* [44].

Let $Q\subset\mathbb{P}^2$ be a (smooth) conic. Then, there exists a line L (resp. a conic R) that is tangent to Q with full multiplicity 2 (resp. 4). In fact we can assume we have determined two distinct intersection points this way. Now, generically, L intersects R at two other points.

Letting $C=L+R$ and $B=3Q$ we have that the pencil generated by B and $2C$ is a Halphen pencil of index two. In particular, blowing-up $\mathbb{P}^2$ at the nine base points $P_1^{(1)},\ldots,P_1^{(3)},P_2^{(1)},\ldots,P_2^{(6)}$ we obtain a rational elliptic surface of index two. And such surface has a type III^* singular fiber.

Example A.2.18 (Two triple lines [54, Example 7.49])**.** *Consider two (distinct) lines*

L_1 and L_2 and let P_3 be their intersection point. Choose a cubic C which intersects L_1 and L_2 at P_3 with multiplicity one and which is tangent to each L_i at a point P_i (with multiplicity two). The pencil $\mathcal{P}$ generated by $B = 3L_1 + 3L_2$ and $2C$ is a Halphen pencil of index two and it yields a fiber of type III^* in the corresponding rational elliptic surface.

Example A.2.19 (A triple line, a double line and another line [54, Example 7.50])**.** *Let C be a smooth cubic. Let L_1 be an inflection line of C at a point P_1 and choose a line L_2 through P_1 which is tangent to C at another point P_2. Let L_3 be any line through P_2 which intersects C at another two points, say P_3 and P_4.*

Then the pencil $\mathcal{P}$ generated by $B = 3L_1 + L_2 + 2L_3$ and $2C$ is a Halphen pencil of index two which yields a fiber of type III^* in the associated rational elliptic surface.

Concretely, we can choose coordinates in $\mathbb{P}^2$ so that C is the cubic given by

$$y^2 z = x(x-z)(x-\alpha\cdot z) \qquad \alpha \in \mathbb{C}\backslash\{0,1\}$$

we can let L_1 be the line $z = 0$ (hence $P_1 = (0:1:0)$) and we can choose L_2 to be either one of the lines $x = 0, x - z = 0$ or $x - \alpha \cdot z = 0$. If we choose L_2 as $x = 0$, then $P_2 = (0:0:1)$ and we can let L_3 be the line $y = 0$.

Example A.2.20 (A triple line, a double line and another line concurrent at a point [54, Example 7.51])**.** *Consider three lines L_1, L_2 and L_3 concurrent at a point P_1 and choose a cubic C so that C is tangent to L_1 at P_1 with full multiplicity, C is tangent to L_2 at a point $P_2 (\neq P_1)$ (with multiplicity two) and it intersects L_3 at two other points P_3 and P_4.*

The pencil $\mathcal{P}$ generated by $B = L_1 + 3L_2 + 2L_3$ and $2C$ is a Halphen pencil of index two and such pencil yields a fiber of type III^* in the associated rational elliptic surface.

Example A.2.21 (A triple line, a conic and a line [54, Example 7.52])**.** *Let Q be a (smooth) conic. Choose a point P_1 in Q and let L_1 be the tangent line to Q at P_1. Choose two other points in Q, say P_2 and P_3, and let L_2 be the line joining them. Let P_4 be the intersection point between L_1 and L_2. We can construct a cubic C through these four points so that C is tangent to Q (resp. L_1) at P_1 with multiplicity four (resp. two).*

The pencil $\mathcal{P}$ generated by $B = 3L_1 + Q + L_2$ and $2C$ is a Halphen pencil of index two which yields a fiber of type III^* in the corresponding rational elliptic surface.

Concretely, choose coordinates in $\mathbb{P}^2$ so that Q is the conic given by $x^2 + yz = 0$ and we have $P_1 = (0 : 1 : 0), P_2 = (-1 : -1 : 1)$ and $P_3 = (0 : 0 : 1)$. Then L_1 is the line $z = 0$, L_2 is the line $x + y = 0, P_4 = (-1 : 1 : 0)$ and C is the cubic given by $(x + z)xz + (x^2 + yz)(x + y) = 0$.

Example A.2.22 (A triple line and a cubic [54, Example 7.53])**.** *Let $D : d = 0$ be a nodal cubic and let P_1 denote its node. Let P_2 be a point in D which is not a flex and let $L : l = 0$ denote the tangent line to D at P_2. Let P_3 be the third intersection point between L and D and let $L' : l' = 0$ denote the line joining P_1 and P_3.*

Then the cubic C given by $l^2l' + d = 0$ is such that the intersection multiplicity of D and C at $P = P_2$ (resp. $P = P_3$) is 4 (resp. 3). Moreover, by construction, the *node P_1 lies in C.*

Concretely, if D is the nodal cubic given by $y^2z = x^2(x+z)$ we have that $P_1 = (0:0:1)$ and we can let $P_2 = (1:0:-1)$ so that L is the line $x+z=0$. Then $P_3 = (0:1:0)$ and L' is the line $x=0$. Thus, C is the cubic given by $z(y^2+x^2+xz)=0$. Note that C consists of a line ($z=0$) and a conic ($y^2+x^2+xz=0$). Moreover, the line is an inflection line of D and the node P_1 lies in the conic.

Now, the pencil $\mathcal{P}$ generated by $B = 3L+D$ and $2C$ is a Halphen pencil of index two and the associated rational elliptic surface has a fiber of type III^*.

Example A.2.23 (A line with multiplicity four and a conic [54, Example 7.54])**.** *Consider either a smooth or nodal cubic C. Choose smooth points $P_1, P_2 \in C$ so that there exists a conic Q which is tangent to C at P_1 (resp. P_2) with multiplicity 4 (resp. 2). Let L be the line joining P_1 and P_2 and let P_3 be the third intersection point between L and C. Then the pencil $\mathcal{P}$ generated by $B = Q+4L$ and $2C$ is a Halphen pencil of index two which yields a fiber of type III^* in the associated rational elliptic surface.*

For instance, consider the cubic C given by $x^2z+(x^2+yz)(y+z)=0$ and let $P_1=(0:1:0)$ and $P_2=(0:0:1)$. Then $L: x=0$ and $P_3=(0:1:-1)$ and we can take $Q: x^2+yz=0$.

Example A.2.24 (A line with multiplicity four and two other lines [54, Example 7.55])**.** *Consider either a smooth or nodal cubic C and let P_4 be a flex point of C. We can always choose two lines L_1 and L_2 through P_4 which are tangent to C at two* *other points P_1 and P_2, respectively. Moreover, if L_3 is the line joining P_1 and P_2, then C intersects L_3 at a third point P_3 and we have that the pencil $\mathcal{P}$ generated by $B = L_1 + L_2 + 4L_3$ and $2C$ is a Halphen pencil of index two with base points*

$$P_1^{(1)}, \ldots, P_1^{(3)}, P_2^{(1)}, \ldots, P_2^{(3)}, P_3^{(1)}, P_3^{(2)}, P_4^{(1)}$$

Blowing-up $\mathbb{P}^2$ at these nine base points yields a fiber of type III^* in the associated rational elliptic surface.

Note that, concretely, we can choose coordinates in $\mathbb{P}^2$ so that C is the cubic given by $y^2z = x(x-z)(x-\alpha \cdot z)$ for some $\alpha \in \mathbb{C}\backslash\{0,1\}$, we can let $P_4 = (0:1:0)$ and we can choose L_1 and L_2 to be the lines $x = 0$ and $x - z = 0$. Then $P_1 = (0:0:1), P_2 = (1:0:1), L_3$ is the line $y = 0$ and $P_3 = (\alpha:0:1)$.

A.2.3 Type II^*

We now construct all possible examples of Halphen pencils of index two that yield a fiber of type II^* in the corresponding rational elliptic surface (Theorem A.1.4).

Example A.2.25 (A triple conic [18], [54, Example 7.56])**.** *We begin with an example of a rational elliptic surface whose Jacobian is the surface X_{211} in Miranda and Persson's list* [44].

Let C be a cubic with a node and let P_0 be an inflection point of C that we take as the identity for the group law. Choose another point P in C satisfying $6P = P_0$. Then there exists a conic Q tangent to C at P with multiplicity 6 and to the pencil *generated by $B = 3Q$ and $2C$ we can associate a rational elliptic fibration $Y \to \mathbb{P}^1$ of index two with $II^* + {}_2I_1 + I_1$ singular fibers.*

Concretely, we blow-up $\mathbb{P}^2$ at the nine points $P_1^{(1)}, \ldots, P_1^{(9)}$ where $P_1^{(1)} = P$. The strict transform of C is the multiple fiber and the strict transform of Q is the component of multiplicity 3 in the II^* fiber that intersects the component of multiplicity 6.

Example A.2.26 (Two triple lines [54, Example 7.57])**.** *Let C be either a smooth or nodal cubic. Let L_1 be an inflection line of C at a point P_1 and let L_2 be a line through P_1 which is tangent to C at another point P_2.*

Then the pencil $\mathcal{P}$ generated by $B = 3L_1 + 3L_2$ and $2C$ is a Halphen pencil of index two which yields a fiber of type II^* in the associated rational elliptic surface.

Concretely, (if C is smooth) we can choose coordinates in $\mathbb{P}^2$ so that C is the cubic given by $y^2z = x(x-z)(x-\alpha\cdot z)$ for some $\alpha \in \mathbb{C}\backslash\{0,1\}$, we can let L_1 be the line $z = 0$ (hence $P_1 = (0:1:0)$) and we can choose L_2 to be either one of the lines $x = 0, x - z = 0$ or $x - \alpha\cdot z = 0$.

If we choose L_2 as $x = 0$, then $P_2 = (0:0:1)$ and, similarly, if we take L_2 as $x - z = 0$ (resp. $x - \alpha\cdot z = 0$), then $P_2 = (1:0:1)$ (resp. $P_2 = (\alpha:0:1)$).

Example A.2.27 (A triple line and a cubic [54, Example 7.58])**.** *Let $D : d = 0$ be a nodal cubic and let P_1 denote its node. Let $L : l = 0$ be an inflection line of D and denote the flex point by P_2. Let $L' : l' = 0$ be the line joining P_1 and P_2.*

Then the cubic C given by $l^2l' + d = 0$ is such that the intersection multiplicity of

D and C at P_2 is 7 and, by construction, the node P_1 lies on it. We also have that L is also an inflection line of C at P_2. Now, the pencil $\mathcal{P}$ generated by $B = D + 3L$ and $2C$ is a Halphen pencil of index two which yields a fiber of type II^* in the associated rational elliptic surface.

Concretely, we can choose as D the nodal cubic given by $y^2z = x^2(x + z)$, then $P_1 = (0 : 0 : 1)$ and we can choose L to be the line $z = 0$ so that $P_2 = (0 : 1 : 0)$. Then L' is the line $x = 0$ and C has equation $z^2x + y^2z - x^3 - x^2z = 0$.

Example A.2.28 (A line with multiplicity four and a conic [54, Example 7.59])**.** *Let C be either a smooth or nodal cubic. Choose a sextactic point $P_1 \in C$ (see Definition A.2.2). And let Q be the corresponding osculating conic at P_1. Choose a line L which is tangent to both Q and C at P_1 and let P_2 be the third point of intersection between L and C. Then the pencil $\mathcal{P}$ generated by $B = Q + 4L$ and $2C$ is a Halphen pencil of index two which yields a fiber of type II^* in the associated rational elliptic surface.*

For instance, consider the cubic C given by

$$-3x^3 + xz^2 + y^2z + 2xy^2 = x^3 + (y^2 - 2x^2 + xz) \cdot (2x + z) = 0$$

Let $P_1 = (0 : 0 : 1)$, let $Q : y^2 - 2x^2 + xz = 0$ and let $L : x = 0$. Then the intersection multiplicity of Q and C at P_1 is 6 and we have that $P_2 = (0 : 1 : 0)$ is a flex point with inflection line $2x + z = 0$.

Example A.2.29 (A line with multiplicity five and another line [54, Example 7.60])**.** *Consider either a smooth or nodal cubic C and let L_1 be an inflection line of C at a point P_1. We can always choose another line L_2 through P_1 which is tangent to C at* *another point P_2. And the pencil $\mathcal{P}$ generated by $B = 5L_2 + L_1$ and $2C$ is a Halphen pencil of index two which yields a fiber of type II^* in the associated rational elliptic surface. Concretely, (if C is smooth) we can choose coordinates in $\mathbb{P}^2$ so that C is the cubic given by $y^2z = x(x-z)(x-\alpha \cdot z)$ for some $\alpha \in \mathbb{C}\backslash\{0,1\}$, we can let L_1 be the line $z = 0$ (hence $P_1 = (0:1:0)$) and we can choose L_2 to be either one of the lines $x = 0, x - z = 0$ or $x - \alpha \cdot z = 0$.*

Appendix B

Non-stable pencils of plane sextics

In Section II.2.5 (and in [56]) we studied the stability of pencils of plane curves of degree six under the action of $SL(3)$ in the sense of geometric invariant theory (GIT). The next paragraphs serve as an appendix to Section II.2.5 (and [56, Section 3]), and provides a complete characterization of the non-stable pencils in $\mathscr{P}_6$ in terms of explicit equations for their generators.

As in Section II.2.5, we use the notation $\langle m_1, \ldots, m_n \rangle$ to denote the subspace of homogeneous polynomials of degree six in the variables x, y and z which is generated by the monomials m_i. Whereas $\rangle m_1, \ldots, m_n \langle$ denotes the subspace of those polynomials which are generated by all the monomials which are different from the m_i.

B.1 Equations associated to non-stability

Given a pencil $\mathcal{P} \in \mathscr{P}_6$ and any of its curves, say C_f, we can represent C_f by a triangle of coefficients of $f = \sum f_{ij} x^i y^j z^{6-i-j}$:

$$\begin{array}{ccccccc}
 & & & f_{60} & & & \\
 & & f_{50} & & f_{51} & & \\
 & & & \vdots & & & \\
 & & & f_{22} & & & \\
 & & & \vdots & & & \\
f_{00} & f_{01} & f_{02} & f_{03} & f_{04} & f_{05} & f_{06}
\end{array}$$

In particular, a pencil $\mathcal{P} \in \mathscr{P}_6$ will satisfy the vanishing conditions in case 1 of Theorem II.2.5.7 if and only if we can find coordinates in $\mathbb{P}^2$ and generators C_f and C_g of $\mathcal{P}$ such that the coefficients below the corresponding lines in one of the cases in Figure B.1 below all vanish.

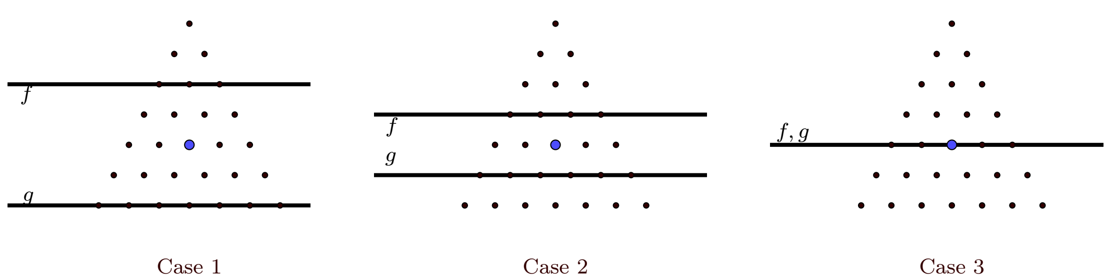


Figure B.1: Pictorial description of Theorem II.2.5.8

This gives a nice visual description of Theorem II.2.5.8 ([56, Theorem 3.1]). Similarly we can prove:

Theorem B.1.1. *A pencil $\mathcal{P} \in P_6$ satisfies the vanishing conditions in case 2 of Theorem II.2.5.7 if and only if there exist coordinates in $\mathbb{P}^2$ and generators C_f and C_g of $\mathcal{P}$ such that either*

1. $f \in \langle x^5z, x^5y, x^6\rangle$ *and g is arbitrary*

2. $f \in \langle x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5\langle$

3. $f \in \langle x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4\langle$

4. $f \in \langle x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3\langle$

5. $f \in \langle x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{33} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, xz^5\langle$

6. $f \in \langle x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{32} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, xz^5, xyz^4\langle$

7. $f \in \langle x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{31} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3\langle$

8. $f \in \langle x^2y^4, x^3z^3, x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{24} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2\langle$

9. $f \in \langle x^2y^3z, x^2y^4, x^3z^3, x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{23} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2, xy^4z, x^2z^4\langle$

10. f *and* $g \in \langle x^2y^2z^2, x^2y^3, x^2y^4, x^iy^jz^{6-i-j}\rangle$, *where* $3 \leq i \leq 6, 0 \leq j \leq 6$ *and* $i + j \leq 6$

Theorem B.1.2. *A pencil* $\mathcal{P} \in P_6$ *satisfies the vanishing conditions in case* 3 *of Theorem II.2.5.7 if and only if there exist coordinates in* $\mathbb{P}^2$ *and generators* C_f *and* C_g *of* $\mathcal{P}$ *such that either*

1. $f \in \langle x^5y, x^6\rangle$ *and* g *is arbitrary*

2. $f \in \langle x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6\langle$

3. $f \in \langle x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5\langle$

4. $f \in \langle x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4\langle$

5. $f \in \langle x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{33} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, xz^5\langle$

6. $f \in \langle x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{40} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, xz^5\langle$

7. $f \in \langle x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{32} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, xz^5, xyz^4\langle$

8. $f \in \langle x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{31} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, xz^5, xyz^4, xy^2z^3\langle$

9. $f \in \langle x^2y^4, x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{24} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2\langle$

10. $f \in \langle x^2y^4, x^3z^3, x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{24} \neq 0$

    and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2, x^2z^4\langle$

11. $f \in \langle x^2y^3z, x^2y^4, x^3z^3, x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{23} \neq 0$

    and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2, x^2z^4\langle$

12. $f \in \langle x^2y^2z^2, x^2y^3, x^2y^4, x^iy^jz^{6-i-j}\rangle$, *where* $3 \leq i \leq 6, 0 \leq j \leq 6$ *and* $i + j \leq 6$, *plus* $f_{22} \neq 0$

    and $g_{00} = \ldots = g_{14} = g_{20} = g_{21} = 0$

13. f *and* $g \in \langle x^2y^2z^2, x^2y^3, x^2y^4, x^iy^jz^{6-i-j}\rangle$, *where* $3 \leq i \leq 6, 0 \leq j \leq 6$ *and* $i + j \leq 6$

Theorem B.1.3. *A pencil* $\mathcal{P} \in \mathscr{P}_6$ *will satisfy the vanishing conditions in case* 4 *of Theorem II.2.5.7 if and only if we can find coordinates in* $\mathbb{P}^2$ *and generators* C_f *and*

C_g of $\mathcal{P}$ such that the coefficients below the corresponding lines in one of the cases in Figure B.2 all vanish.

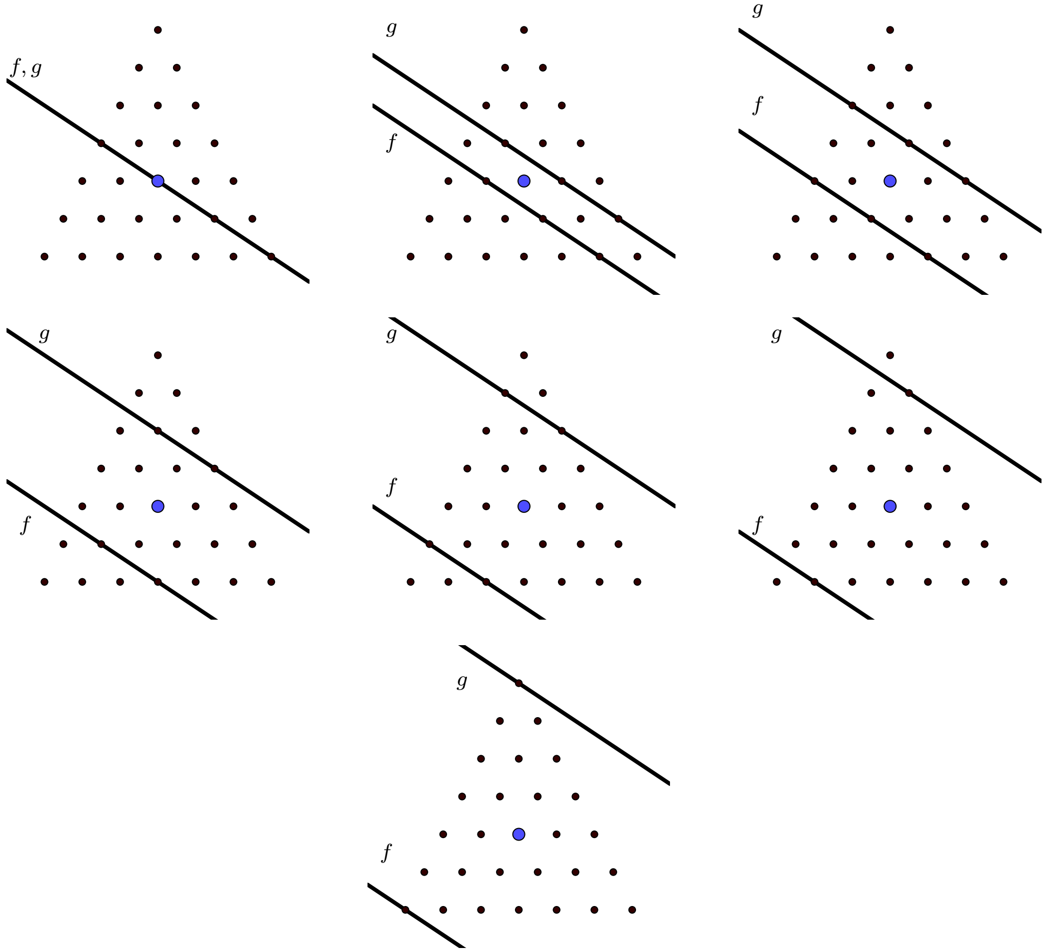


Figure B.2: Pictorial description of case 4 of Theorem II.2.5.7

Theorem B.1.4. *A pencil $\mathcal{P} \in P_6$ satisfies the vanishing conditions in case 5 of Theorem II.2.5.7 if and only if there exist coordinates in $\mathbb{P}^2$ and generators C_f and C_g of $\mathcal{P}$ such that either*

1. $f \in \langle x^6 \rangle$

and $g \in \rangle z^6 \langle$

2. $f \in \langle x^5y, x^6 \rangle$

   and $g \in \rangle z^6, yz^5 \langle$

3. $f \in \langle x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$*, with* $f_{41} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, xz^5, xyz^4 \langle$

4. $f \in \langle x^4y^2, x^5z, x^5y, x^6 \rangle$*, with* $f_{42} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, xz^5 \langle$

5. $f \in \langle x^5z, x^5y, x^6 \rangle$*, with* $f_{50} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, xz^5 \langle$

6. $f \in \langle x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3 \langle$*, with* $m_{ijkl} = 0$ *for* i, j, k *and* l *(in order) in the list below:*

   $$\{1,0,3,2\}, \{1,0,3,3\}, \{1,0,4,0\}, \{1,0,4,1\}, \{1,0,4,2\}, \{1,0,5,0\},$$
   $$\{1,1,3,2\}, \{1,1,4,0\}, \{1,1,4,1\}, \{1,2,4,0\}, \{2,0,3,2\}, \{2,0,4,0\}$$

7. $f \in \langle x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2 \langle$*, with* $m_{ijkl} = 0$ *for* i, j, k *and* l *(in order)* *in the list below:*

$$\{1,0,3,1\},\{1,0,3,2\},\{1,0,3,3\},\{1,0,4,0\},\{1,0,4,1\},\{1,0,4,2\},$$
$$\{1,0,5,0\},\{1,1,3,1\},\{1,1,3,2\},\{1,1,4,0\},\{1,1,4,1\},\{1,2,3,1\},$$
$$\{1,2,4,0\},\{2,0,3,1\},\{2,0,3,2\},\{2,0,4,0\},\{2,1,3,1\}$$

8. $f \in \langle x^2yz^3, x^2y^2z^2, x^2y^3, x^2y^4, x^iy^jz^{6-i-j}\rangle$, *where* $3 \leq i \leq 6, 0 \leq j \leq 6$ *and* $i+j \leq 6$

and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z\langle$, *with* $m_{ijkl} = 0$ *for* i,j,k *and* l *(in order) in the list below:*

$$\{1,0,2,1\},\{1,0,2,2\},\{1,0,2,3\},\{1,0,2,4\},\{1,0,3,0\},\{1,0,3,1\},$$
$$\{1,0,3,2\},\{1,0,3,3\},\{1,0,4,0\},\{1,0,4,1\},\{1,0,4,2\},\{1,0,5,0\},$$
$$\{1,1,2,1\},\{1,1,2,2\},\{1,1,2,3\},\{1,1,2,4\},\{1,1,3,0\},\{1,1,3,1\},$$
$$\{1,1,3,2\},\{1,1,4,0\},\{1,1,4,1\},\{1,2,2,1\},\{1,2,2,2\},\{1,2,2,3\},$$
$$\{1,2,3,0\},\{1,2,3,1\},\{1,2,4,0\},\{1,3,2,1\},\{1,3,2,2\},\{1,3,3,0\},$$
$$\{1,4,2,1\},\{2,0,2,1\},\{2,0,2,2\},\{2,0,2,3\},\{2,0,3,0\},\{2,0,3,1\},$$
$$\{2,0,3,2\},\{2,0,4,0\},\{2,1,2,2\},\{2,1,3,0\},\{2,1,3,1\},\{2,2,3,0\}$$

9. $f \in \langle x^2y^2z^2, x^2y^3, x^2y^4, x^iy^jz^{6-i-j}\rangle$, *where* $3 \leq i \leq 6, 0 \leq j \leq 6$ *and* $i+j \leq 6$

and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2\langle$, *with* $m_{ijkl} = 0$ *for* i,j,k *and* l *(in order) in the list below:*

$\{2,0,2,2\},\{2,0,2,3\},\{2,0,3,0\},\{2,0,3,1\},\{2,0,3,2\},\{2,0,4,0\},$

$\{2,1,2,2\},\{2,1,3,0\},\{2,1,3,1\},\{2,2,3,0\}$

Theorem B.1.5. *A pencil $\mathcal{P} \in P_6$ satisfies the vanishing conditions in case 6 of Theorem II.2.5.7 if and only if there exist coordinates in $\mathbb{P}^2$ and generators C_f and C_g of $\mathcal{P}$ such that either*

1. $f \in \langle x^5y, x^6\rangle$, *with* $f_{51} \neq 0$

   and $g \in \rangle z^6, yz^5, xz^5\langle$

2. $f \in \langle x^6\rangle$

   and $g \in \rangle z^6, yz^5\langle$

3. $f \in \langle x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4\langle$, *with* $m_{ijkl} = 0$ *for* i, j, k *and* l *(in order) in the list below:*

   $\{1,0,4,1\},\{1,0,4,2\},\{1,0,5,0\},\{1,0,5,1\},\{1,1,4,1\},\{1,1,5,0\},$

   $\{2,0,4,1\}$

4. $f \in \langle x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3\langle$, *with* $m_{ijkl} = 0$ *for* i, j, k *and* l *(in order) in the list below:*

$\{1,0,3,2\},\{1,0,3,3\},\{1,0,4,0\},\{1,0,4,1\},\{1,0,4,2\},\{1,0,5,0\},$

$\{1,0,5,1\},\{1,1,3,2\},\{1,1,4,0\},\{1,1,4,1\},\{1,1,5,0\},\{1,2,4,0\},$

$\{2,0,3,2\},\{2,0,4,0\},\{2,0,4,1\},\{2,1,4,0\},$

5. *$f \in \langle x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$*

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^4\langle$, with $m_{ijkl} = 0$ for i, j, k and l (in order) in the list below:

$\{1,0,3,1\},\{1,0,3,2\},\{1,0,3,3\},\{1,0,4,0\},\{1,0,4,1\},\{1,0,4,2\},$

$\{1,0,5,0\},\{1,0,5,1\},\{1,1,3,1\},\{1,1,3,2\},\{1,1,4,0\},\{1,1,4,1\},$

$\{1,1,5,0\},\{1,2,3,1\},\{1,2,4,0\},\{2,0,3,1\},\{2,0,3,2\},\{2,0,4,0\},$

$\{2,0,4,1\},\{2,1,3,1\},\{2,1,4,0\},\{3,0,3,1\}$

6. *$f \in \langle x^2yz^3, x^2y^2z^2, x^2y^3, x^2y^4, x^iy^jz^{6-i-j}\rangle$, where $3 \le i \le 6, 0 \le j \le 6$ and $i + j \le 6$*

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z\langle$, with $m_{ijkl} = 0$ for i, j, k and l (in order) in the list below:

$\{1,0,2,1\},\{1,0,2,2\},\{1,0,2,3\},\{1,0,2,4\},\{1,0,3,0\},\{1,0,3,1\},$

$\{1,0,3,2\},\{1,0,3,3\},\{1,0,4,0\},\{1,0,4,1\},\{1,0,4,2\},\{1,0,5,0\},$

$\{1,0,5,1\},\{1,1,2,1\},\{1,1,2,2\},\{1,1,2,3\},\{1,1,2,4\},\{1,1,3,0\},$

$\{1,1,3,1\},\{1,1,3,2\},\{1,1,4,0\},\{1,1,4,1\},\{1,1,5,0\},\{1,2,2,1\},$

$\{1,2,2,2\},\{1,2,2,3\},\{1,2,3,0\},\{1,2,3,1\},\{1,2,4,0\},\{1,3,2,1\},$

$\{1,3,2,2\},\{1,3,3,0\},\{1,4,2,1\},\{2,0,2,1\},\{2,0,2,2\},\{2,0,2,3\},$

$\{2,0,3,0\},\{2,0,3,1\},\{2,0,3,2\},\{2,0,4,0\},\{2,0,4,1\},\{2,1,2,2\},$

$\{2,1,3,0\},\{2,1,3,1\},\{2,1,4,0\},\{2,2,3,0\},\{3,0,3,1\}$

7. $f \in \langle x^2y^2z^2, x^2y^3, x^2y^4, x^iy^jz^{6-i-j}\rangle$, *where* $3 \le i \le 6, 0 \le j \le 6$ *and* $i+j \le 6$ *and* $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2\langle$, *with* $m_{ijkl} = 0$ *for* i, j, k *and* l *(in order) in the list below:*

$\{2,0,2,2\},\{2,0,2,3\},\{2,0,3,0\},\{2,0,3,1\},\{2,0,3,2\},\{2,0,4,0\},$

$\{2,0,4,1\},\{2,1,2,2\},\{2,1,3,0\},\{2,1,3,1\},\{2,1,4,0\},\{2,2,3,0\},$

$\{3,0,3,1\}$

8. $f \in \langle x^4y^2, x^5z, x^5y, x^6\rangle$ *and* $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2, xy^4z, xy^5\langle$

Theorem B.1.6. *A pencil $\mathcal{P} \in \mathscr{P}_6$ will satisfy the vanishing conditions in case 7 of Theorem II.2.5.7 if and only if we can find coordinates in $\mathbb{P}^2$ and generators C_f and C_g of $\mathcal{P}$ such that the coefficients on the left of the corresponding lines in one of the cases in Figure B.3 all vanish.*

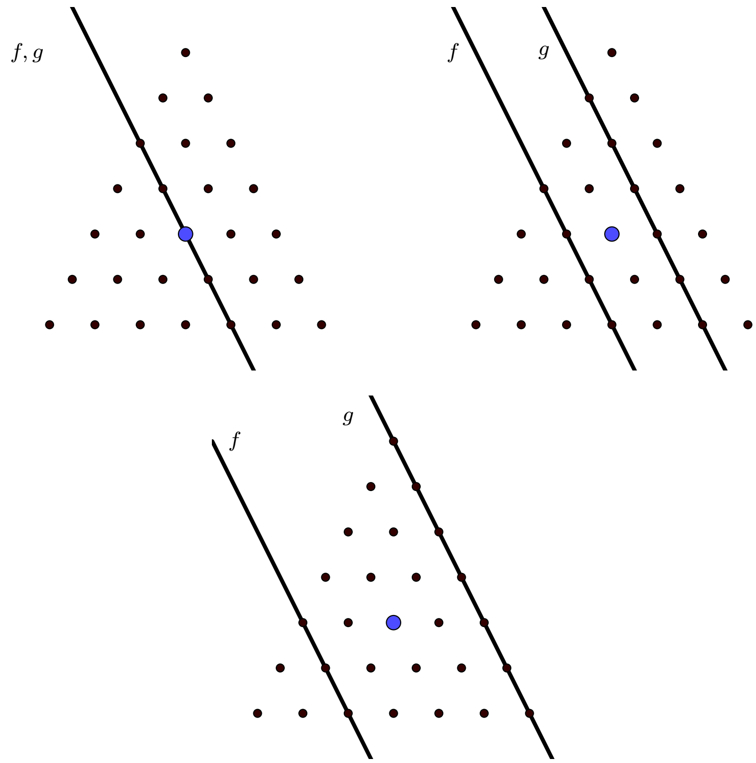


Figure B.3: Pictorial description of case 7 of Theorem II.2.5.7

B.2 Equations associated to unstability

Theorem B.2.1. *A pencil $\mathcal{P} \in P_6$ satisfies the vanishing conditions in case 1 of Theorem II.2.5.6 if and only if there exist coordinates in $\mathbb{P}^2$ and generators C_f and C_g of $\mathcal{P}$ such that either*

1. $f \in \langle x^5z, x^5y, x^6\rangle$ *and* g *is arbitrary*

2. $f \in \langle x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4\langle$

3. $f \in \langle x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3\langle$

4. $f \in \langle x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2\langle$

5. $f \in \langle x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{33} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, xz^5\langle$

6. $f \in \langle x^3yz, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3\langle$

7. $f \in \langle x^2y^4, x^3z^3, x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{24} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2, xy^4z, x^2z^4\langle$

8. $f \in \langle x^3z^3, x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2, xy^4z\langle$

Theorem B.2.2. *A pencil $\mathcal{P} \in P_6$ satisfies the vanishing conditions in case 2 of Theorem II.2.5.6 if and only if there exist coordinates in $\mathbb{P}^2$ and generators C_f and C_g of $\mathcal{P}$ such that either*

1. $f \in \langle x^5y, x^6 \rangle$ *and* g *is arbitrary*

2. $f \in \langle x^5z, x^5y, x^6 \rangle$

   and $g \rangle z^6 \langle$

3. $f \in \langle x^4y^2, x^5z, x^5y, x^6 \rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4 \langle$

4. $f \in \langle x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3 \langle$

5. $f \in \langle x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$, *with* $f_{33} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, xz^5, xyz^4 \langle$

6. $f \in \langle x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$, *with* $f_{40} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, xz^5 \langle$

7. $f \in \langle x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$, *with* $f_{32} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, xz^5, xyz^4, xy^2z^3 \langle$

8. $f \in \langle x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2 \langle$

9. $f \in \langle x^2y^3z, x^2y^4, x^3z^3, x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2, xy^4z \langle$, *with* $m_{ijkl} \neq 0$ *for* i, j, k *and* l *(in order) in the list below:*

$$\{2,0,2,3\},\{2,0,2,4\},\{2,0,3,0\},\{2,1,2,3\}$$

Theorem B.2.3. *A pencil $\mathcal{P} \in P_6$ satisfies the vanishing conditions in case 3 of Theorem II.2.5.6 if and only if there exist coordinates in $\mathbb{P}^2$ and generators C_f and C_g of $\mathcal{P}$ such that either*

1. $f \in \langle x^6 \rangle$ *and g is arbitrary*
2. $f \in \langle x^5y, x^6 \rangle$

   and $g \in \rangle z^6 \langle$
3. $f \in \langle x^5z, x^5y, x^6 \rangle$

   and $g \in \rangle z^6, yz^5 \langle$
4. $f \in \langle x^4y^2, x^5z, x^5y, x^6 \rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4 \langle$
5. $f \in \langle x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$, *with* $f_{41} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, xz^5 \langle$
6. $f \in \langle x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2 \langle$, *with $m_{ijkl} \neq 0$ for i, j, k and l (in order) in the list below:*

$$\{1,0,3,2\},\{1,0,3,3\},\{1,0,4,0\},\{1,0,4,1\},\{1,1,3,2\},\{1,1,3,3\},$$

$$\{1,1,4,0\},\{1,2,3,2\}$$

7. $f \in \langle x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z\langle$, *with* $m_{ijkl} \neq 0$ *for* i, j, k *and* l *(in order) in the list below:*

$$\{1,0,3,1\},\{1,0,3,2\},\{1,0,3,3\},\{1,0,4,0\},\{1,0,4,1\},\{1,1,3,1\},$$

$$\{1,1,3,2\},\{1,1,3,3\},\{1,1,4,0\},\{1,2,3,1\},\{1,2,3,2\},\{1,3,3,1\},$$

$$\{2,0,3,1\}$$

8. $f \in \langle x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4\langle$

9. $f \in \langle x^2y^2z^2, x^2y^3, x^2y^4, x^iy^jz^{6-i-j}\rangle$, *where* $3 \leq i \leq 6, 0 \leq j \leq 6$ *and* $i + j \leq 6$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2, xy^4z\langle$, *with* $m_{ijkl} \neq 0$ *for* i, j, k *and* l *(in order) in the list below:*

$$\{2,0,2,2\},\{2,0,2,3\},\{2,0,2,4\},\{2,0,3,0\},\{2,0,3,1\},\{2,1,2,2\},$$

$$\{2,1,2,3\},\{2,1,3,0\}$$

Theorem B.2.4. *A pencil $\mathcal{P} \in P_6$ satisfies the vanishing conditions in case 4 of Theorem II.2.5.6 if and only if there exist coordinates in $\mathbb{P}^2$ and generators C_f and C_g of $\mathcal{P}$ such that either*

1. $f \in \langle x^6 \rangle$

   and $g \in \rangle z^6 \langle$

2. $f \in \langle x^5y, x^6 \rangle$

   and $g \in \rangle z^6, yz^5 \langle$

3. $f \in \langle x^4y^2, x^5z, x^5y, x^6 \rangle$, *with* $f_{42} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, xz^5 \langle$

4. $f \in \langle x^5z, x^5y, x^6 \rangle$, *with* $f_{50} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, xz^5 \langle$

5. $f \in \langle x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$, *with* $f_{41} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, xz^5, xyz^4 \langle$

6. $f \in \langle x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2 \langle$, *with* $m_{ijkl} \neq 0$ *for* i, j, k *and* l *(in order) in the list below:*

$\{1,0,3,1\},\{1,0,3,2\},\{1,0,3,3\},\{1,0,4,0\},\{1,0,4,1\},\{1,0,4,2\},$

$\{1,0,5,0\},\{1,1,3,1\},\{1,1,3,2\},\{1,1,3,3\},\{1,1,4,0\},\{1,1,4,1\},$

$\{1,2,3,1\},\{1,2,3,2\},\{1,2,4,0\},\{1,3,3,1\},\{2,0,3,1\},\{2,0,3,2\},$

$\{2,0,4,0\},\{2,1,3,1\}$

7. *$f \in \langle x^2y^2z^2, x^2y^3, x^2y^4, x^iy^jz^{6-i-j}\rangle$, where $3 \le i \le 6, 0 \le j \le 6$ and $i+j \le 6$ and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z\langle$, with $m_{ijkl} \neq 0$ for i,j,k and l (in order) in the list below:*

$\{1,0,2,2\},\{1,0,2,3\},\{1,0,2,4\},\{1,0,3,0\},\{1,0,3,1\},\{1,0,3,2\},$

$\{1,0,3,3\},\{1,0,4,0\},\{1,0,4,1\},\{1,0,4,2\},\{1,0,5,0\},\{1,1,2,2\},$

$\{1,1,2,3\},\{1,1,2,4\},\{1,1,3,0\},\{1,1,3,1\},\{1,1,3,2\},\{1,1,3,3\},$

$\{1,1,4,0\},\{1,1,4,1\},\{1,2,2,2\},\{1,2,2,3\},\{1,2,3,0\},\{1,2,3,1\},$

$\{1,2,3,2\},\{1,2,4,0\},\{1,3,2,2\},\{1,3,3,0\},\{1,3,3,1\},\{1,4,3,0\},$

$\{2,0,2,2\},\{2,0,2,3\},\{2,0,2,4\},\{2,0,3,0\},\{2,0,3,1\},\{2,0,3,2\},$

$\{2,0,4,0\},\{2,1,2,2\},\{2,1,2,3\},\{2,1,3,0\},\{2,1,3,1\},\{2,2,3,0\}$

8. *$f \in \langle x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, with $f_{32} \neq 0$ and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, x^2z^4\langle$*

9. $f \in \langle x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6 \rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3 \langle$ *plus* $m_{2040} = 0$

10. $f \in \langle x^2yz^3, x^2y^2z^2, x^2y^3, x^2y^4, x^iy^jz^{6-i-j} \rangle$, *where* $3 \leq i \leq 6, 0 \leq j \leq 6$ *and* $i + j \leq 6$

    and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3, xy^3z^2, xy^4z \langle$, *with* $m_{ijkl} \neq 0$ *for* i, j, k *and* l *(in order) in the list below:*

$$\{2,0,2,1\}, \{2,0,2,2\}, \{2,0,2,3\}, \{2,0,2,4\}, \{2,0,3,0\}, \{2,0,3,1\},$$
$$\{2,0,3,2\}, \{2,0,4,0\}, \{2,1,2,2\}, \{2,1,2,3\}, \{2,1,3,0\}, \{2,1,3,1\},$$
$$\{2,2,3,0\}$$

Theorem B.2.5. *A pencil* $\mathcal{P} \in P_6$ *satisfies the vanishing conditions in case* 5 *of Theorem II.2.5.6 if and only if there exist coordinates in* $\mathbb{P}^2$ *and generators* C_f *and* C_g *of* $\mathcal{P}$ *such that either*

1. $f \in \langle x^5y, x^6 \rangle$, *with* $f_{51} \neq 0$

   and $g \in \rangle z^6, yz^5, xz^5 \langle$

2. $f \in \langle x^6 \rangle$

   and $g \in \rangle z^6, yz^5 \langle$

3. $f \in \langle x^4y^2, x^5z, x^5y, x^6 \rangle$

and $g \in \rangle z^6, yz^5, y^2z^4\langle$, *with* $m_{ijkl} \neq 0$ *for* i, j, k *and* l *(in order) in the list below:*

$$\{1,0,4,2\}, \{1,0,5,0\}, \{1,0,5,1\}, \{1,1,5,0\}$$

4. $f \in \langle x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$, *with* $f_{41} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, xz^5, xyz^4, x^2z^4\langle$

5. $f \in \langle x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2\langle$, *with* $m_{ijkl} \neq 0$ *for* i, j, k *and* l *(in order) in the list below:*

$$\{1,0,3,1\}, \{1,0,3,2\}, \{1,0,3,3\}, \{1,0,4,0\}, \{1,0,4,1\}, \{1,0,4,2\},$$
$$\{1,0,5,0\}, \{1,0,5,1\}, \{1,1,3,1\}, \{1,1,3,2\}, \{1,1,3,3\}, \{1,1,4,0\},$$
$$\{1,1,4,1\}, \{1,1,5,0\}, \{1,2,3,1\}, \{1,2,3,2\}, \{1,2,4,0\}, \{1,3,3,1\},$$
$$\{2,0,3,1\}, \{2,0,3,2\}, \{2,0,4,0\}, \{2,0,4,1\}, \{2,1,3,1\}, \{2,1,4,0\},$$
$$\{3,0,3,1\}$$

6. $f \in \langle x^2yz^3, x^2y^2z^2, x^2y^3, x^2y^4, x^iy^jz^{6-i-j}\rangle$, *where* $3 \leq i \leq 6, 0 \leq j \leq 6$ *and* $i + j \leq 6$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z\langle$, *with* $m_{ijkl} \neq 0$ *for* i, j, k *and* l *(in order) in the list below:*

$$\{1,0,2,1\},\{1,0,2,2\},\{1,0,2,3\},\{1,0,2,4\},\{1,0,3,0\},\{1,0,3,1\},$$
$$\{1,0,3,2\},\{1,0,3,3\},\{1,0,4,0\},\{1,0,4,1\},\{1,0,4,2\},\{1,0,5,0\},$$
$$\{1,0,5,1\},\{1,1,2,1\},\{1,1,2,2\},\{1,1,2,3\},\{1,1,2,4\},\{1,1,3,0\},$$
$$\{1,1,3,1\},\{1,1,3,2\},\{1,1,3,3\},\{1,1,4,0\},\{1,1,4,1\},\{1,1,5,0\},$$
$$\{1,2,2,1\},\{1,2,2,2\},\{1,2,2,3\},\{1,2,3,0\},\{1,2,3,1\},\{1,2,3,2\},$$
$$\{1,2,4,0\},\{1,3,2,1\},\{1,3,2,2\},\{1,3,3,0\},\{1,3,3,1\},\{1,4,2,1\},$$
$$\{1,4,3,0\},\{2,0,2,1\},\{2,0,2,2\},\{2,0,2,3\},\{2,0,2,4\},\{2,0,3,0\},$$
$$\{2,0,3,1\},\{2,0,3,2\},\{2,0,4,0\},\{2,0,4,1\},\{2,1,2,2\},\{2,1,2,3\},$$
$$\{2,1,3,0\},\{2,1,3,1\},\{2,1,4,0\},\{2,2,3,0\},\{3,0,3,1\}$$

7. $f \in \langle x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3\langle$, *with* $m_{ijkl} \neq 0$ *for* i, j, k *and* l *(in order) in the list below:*

$$\{2,0,3,2\},\{2,0,4,0\},\{2,0,4,1\},\{2,1,4,0\},\{3,0,3,1\}$$

Theorem B.2.6. *A pencil* $\mathcal{P} \in P_6$ *satisfies the vanishing conditions in case* 6 *of Theorem II.2.5.6 if and only if there exist coordinates in* $\mathbb{P}^2$ *and generators* C_f *and* C_g *of* $\mathcal{P}$ *such that either*

1. $f \in \langle x^5y, x^6\rangle$

and $g \in \rangle z^6, yz^5, xz^5\langle$

2. $f \in \langle x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4,$ *with* $m_{ijkl} \neq 0$ *for* i, j, k *and* l *(in order) in the list below:*

$$\{1,0,4,2\}, \{1,0,5,0\}, \{1,0,5,1\}, \{1,0,6,0\}, \{1,1,5,0\}, \{2,0,5,0\}$$

3. $f \in \langle x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle,$ *with* $f_{41} \neq 0$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, xz^5, xyz^4, x^2z^4$

4. $f \in \langle x^3yz^2, x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2\langle,$ *with* $m_{ijkl} \neq 0$ *for* i, j, k *and* l *(in order) in the list below:*

$$\{1,0,3,1\}, \{1,0,3,2\}, \{1,0,3,3\}, \{1,0,4,0\}, \{1,0,4,1\}, \{1,0,4,2\},$$
$$\{1,0,5,0\}, \{1,0,5,1\}, \{1,0,6,0\}, \{1,1,3,1\}, \{1,1,3,2\}, \{1,1,3,3\},$$
$$\{1,1,4,0\}, \{1,1,4,1\}, \{1,1,5,0\}, \{1,2,3,1\}, \{1,2,3,2\}, \{1,2,4,0\},$$
$$\{1,3,3,1\}, \{2,0,3,1\}, \{2,0,3,2\}, \{2,0,4,0\}, \{2,0,4,1\}, \{2,0,5,0\},$$
$$\{2,1,3,1\}, \{2,1,4,0\}, \{3,0,3,1\}, \{3,0,4,0\}$$

5. $f \in \langle x^2yz^3, x^2y^2z^2, x^2y^3, x^2y^4, x^iy^jz^{6-i-j}\rangle,$ *where* $3 \leq i \leq 6, 0 \leq j \leq 6$ *and* $i + j \leq 6$

and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z\langle$, *with* $m_{ijkl} \neq 0$ *for* i, j, k *and* l *(in order) in the list below:*

$$\{1,0,2,1\},\{1,0,2,2\},\{1,0,2,3\},\{1,0,2,4\},\{1,0,3,0\},\{1,0,3,1\},$$
$$\{1,0,3,2\},\{1,0,3,3\},\{1,0,4,0\},\{1,0,4,1\},\{1,0,4,2\},\{1,0,5,0\},$$
$$\{1,0,5,1\},\{1,0,6,0\},\{1,1,2,1\},\{1,1,2,2\},\{1,1,2,3\},\{1,1,2,4\},$$
$$\{1,1,3,0\},\{1,1,3,1\},\{1,1,3,2\},\{1,1,3,3\},\{1,1,4,0\},\{1,1,4,1\},$$
$$\{1,1,5,0\},\{1,2,2,1\},\{1,2,2,2\},\{1,2,2,3\},\{1,2,3,0\},\{1,2,3,1\},$$
$$\{1,2,3,2\},\{1,2,4,0\},\{1,3,2,1\},\{1,3,2,2\},\{1,3,3,0\},\{1,3,3,1\},$$
$$\{2,0,2,1\},\{2,0,2,2\},\{2,0,2,3\},\{2,0,2,4\},\{2,0,3,0\},\{2,0,3,1\},$$
$$\{2,0,3,2\},\{2,0,4,0\},\{2,0,4,1\},\{2,0,5,0\},\{2,1,2,2\},\{2,1,2,3\},$$
$$\{2,1,3,0\},\{2,1,3,1\},\{2,1,4,0\},\{2,2,3,0\},\{3,0,3,1\},\{3,0,4,0\}$$

6. $f \in \langle x^3y^2z, x^3y^3, x^4z^2, x^4yz, x^4y^2, x^5z, x^5y, x^6\rangle$

   and $g \in \rangle z^6, yz^5, y^2z^4, y^3z^3, y^4z^2, y^5z, y^6, xz^5, xyz^4, xy^2z^3\langle$, *with* $m_{ijkl} \neq 0$ *for* i, j, k *and* l *(in order) in the list below:*

$$\{2,0,3,2\},\{2,0,4,0\},\{2,0,4,1\},\{2,0,5,0\},\{2,1,4,0\},\{3,0,3,1\},$$
$$\{3,0,4,0\}$$

Bibliography

[1] V. Alexeev and V. V. Nikulin, *Chapter 2. general theory of dpn surfaces and k3 surfaces with non-symplectic involution*, MSJ Memoirs, vol. Volume 15, The Mathematical Society of Japan, Tokyo, Japan, 2006.

[2] K. Ascher and D. Bejleri, *Log canonical models of elliptic surfaces*, Advances in Mathematics **320** (2017), 210–243.

[3] ______, *Moduli of weighted stable elliptic surfaces and invariance of log plurigenera*, 2018.

[4] A. Beauville, *Complex algebraic surfaces*, London Mathematical Society Student Texts, Cambridge University Press, 1996.

[5] J. Blanc, I. Pan, and T. Vust, *On birational transformations of pairs in the complex plane*, Geometriae Dedicata **139** (2009Apr), no. 1, 57–73.

[6] F. Bogomolov and Y. Tschinkel, *Density of rational points on elliptic K3 surfaces*, Asian Journal of Mathematics **4** (1999).

[7] E. Bombieri and J. Bourgain, *A problem on sums of two squares*, Int. Math. Res. Not. IMRN **11** (2015), 3343–3407.

[8] S. Cantat and I. Dolgachev, *Rational surfaces with a large group of automorphisms*, Journal of the American Mathematical Society **25** (2012), no. 3, 863–905.

[9] A. Cayley, *On the conic of five-pointic contact at any point of a plane curve*, Philosophical Transactions of the Royal Society of London **149** (1859), 371–400.

[10] ______, *On the sextactic points of a plane curve*, Philosophical Transactions of the Royal Society of London **155** (1865), 545–578.

[11] I. Cheltsov, *Worst singularities of plane curves of given degree*, The Journal of Geometric Analysis **27** (2017Jul), no. 3, 2302–2338.

[12] I. Cheltsov and I. Karzhemanov, *Halphen pencils on quartic threefolds*, Adv. Math. **223** (2010), no. 2, 594–618.

[13] I. Cheltsov and J. Park, *Halphen pencils on weighted Fano threefold hypersurfaces*, Cent. Eur. J. Math. **7** (2009), no. 1, 1–45.

[14] D. Abramovich and L.Y. Fong and J. Kollár and J. McKernan, *Semi log canonical surfaces*, Astérisque **211** (1992), 139–154.

[15] I. Dolgachev, *Lectures on invariant theory*, London Mathematical Society Lecture Note Series, Cambridge University Press, 2003.

[16] I. Dolgachev and F. Cossec, *Enriques surfaces i*, Progress in Mathematics, vol. 76, Birkhäuser, 1989.

[17] Y. Fujimoto, *On rational elliptic surfaces with multiple fibers*, Publications of the Research Institute for Mathematical Sciences **26** (1990), 1–13.

[18] ______, *On explicit constructions of rational elliptic surfaces with multiple fibers*, Journal of Mathematics of Kyoto University **38** (1998), no. 3, 517–523.

[19] W. Fulton, *Introduction to intersection theory in algebraic geometry*, CBMS Regional Conference Series in Mathematics, vol. 54, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1984.

[20] D. Fusi, *Construction of linear pencils of cubic curves with Mordell-Weil rank six and seven*, Comment. Math. Univ. St. Pauli **55** (2006), no. 2, 195–205.

[21] A. Grassi and T. Weigand, *On topological invariants of algebraic threefolds with ($\mathbb{Q}$-factorial) singularities*, arXiv e-prints (April 2018), available at arXiv:1804.02424.

[22] M. Gross, *Elliptic Threefolds II: Multiple Fibers*, Transactions of the AMS **349** (1997), 3409–3468.

[23] P. Hacking, *Compact moduli of plane curves*, Duke Math. J. **124** (200408), no. 2, 213–257.

[24] G-H Halphen, *Sur les courbes planes du sixième degré à neuf points doubles*, Bulletin de la Société Mathématique de France **10** (1882), 162–172.

[25] B. Harbourne and W. E. Lang, *Multiple fibers on rational elliptic surfaces*, Transactions of the American Mathematical Society **307** (1988), no. 1, 205–223.

[26] D. Huybrechts, *Lectures on K3 Surfaces*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2016.

[27] H. Kim and Y. Lee, *Log canonical thresholds of semistable plane curves*, Mathematical Proceedings of the Cambridge Philosophical Society **137** (2004), no. 2.

[28] Y. Kimura, *Gauge groups and matter fields on some models of F-theory without section*, Journal of High Energy Physics (2016Mar).

[29] ______, *K3 surfaces without section as double covers of Halphen surfaces, and F-theory compactifications*, Progress of Theoretical and Experimental Physics **2018** (2018), no. 4.

[30] ______, *$SU(n) \times \mathbb{Z}_2$ in F-theory on K3 surfaces without section as double covers of Halphen surfaces*, 2018.

[31] R. Kloosterman, *Classification of all Jacobian elliptic fibrations on certain K3 surfaces*, Journal of the Mathematical Society of Japan **58** (200607), no. 3, 665–680.

[32] K. Kodaira, *On the structure of compact complex analytic surfaces, i*, American Journal of Mathematics **86** (1964), no. 4, 751–798.

[33] ———, *On the structure of compact complex analytic surfaces, ii*, American Journal of Mathematics **88** (1966), no. 3, 682–721.

[34] J. Kollár, *Polynomials with integral coefficients, equivalent to a given polynomial.*, Electronic Research Announcements of the American Mathematical Society [electronic only] **3** (1997), no. 3, 17–27.

[35] ———, *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, 1997, pp. 221–287.

[36] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 2008.

[37] J. Kollár and N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Inventiones mathematicae **91** (1988Jun), no. 2, 299–338.

[38] J. Kollár, K. E. Smith, and A. Corti, *Rational and nearly rational varieties*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2004.

[39] M. Kuwata, *Elliptic fibrations on quartic K3 surfaces with large Picard numbers*, Pacific Journal of Mathematics **171** (199511).

[40] R. Miranda, *On the stability of pencils of cubic curves*, American Journal of Mathematics **102** (1980), no. 6, 1177–1202.

[41] ———, *Smooth models for elliptic threefolds*, The birational geometry of degenerations, 1983.

[42] ———, *The basic theory of elliptic surfaces: notes of lectures*, ETS Editrice, 1989.

[43] ———, *Persson's list of singular fibers for a rational elliptic surface.*, Mathematische Zeitschrift **205** (1990), no. 2, 191–212.

[44] R. Miranda and U. Persson, *On extremal rational elliptic surfaces*, Mathematische Zeitschrift **193** (1986Dec), no. 4, 537–558.

[45] A. Néron, *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux*, Publications Mathématiques de l'IHÉS **21** (1964), 5–128.

[46] V. Pastro, *Construction of rational elliptic surfaces with Mordell-Weil rank four*, Comment. Math. Univ. St. Pauli **61** (2012), no. 1, 29–42.

[47] U. Persson, *Configurations of Kodaira fibers on rational elliptic surfaces*, Math. Z. **205** (1990), no. 1, 1–47.

[48] S. Keel and K. Matsuki and J. McKernan, *Log abundance theorem for threefolds*, Duke Mathematical Journal **75** (199407), no. 1, 99–119.

[49] H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. **220** (2001), no. 1, 165–229.

[50] C. Salgado, *Construction of linear pencils of cubics with Mordell-Weil rank five*, Comment. Math. Univ. St. Pauli **58** (2009), no. 2, 95–104.

[51] I.R. Shafarevich, V.I. Danilov, R. Treger, and V.A. Iskovskikh, *Algebraic geometry ii: Cohomology of algebraic varieties. algebraic surfaces*, Encyclopaedia of Mathematical Sciences, Springer Berlin Heidelberg, 2013.

[52] J. Shah, *A Complete Moduli Space for K3 Surfaces of Degree 2*, Annals of Mathematics **112** (1980), no. 3, 485–510.

[53] I. Shimada, *On elliptic K3 surfaces*, The Michigan Mathematical Journal **47** (2000), no. 3, 423–446.

[54] A. Zanardini, *Explicit Constructions of Halphen Pencils*, arXiv e-prints (August 2020), available at arXiv:2008.08128.

[55] ______, *Stability of pencils of plane curves, log canonical thresholds and multiplicities*, arXiv e-prints (January 2021), available at arXiv:2101.01756.

[56] ———, *Stability of pencils of plane sextics and Halphen pencils of index two*, arXiv e-prints (January 2021), available at arXiv:2101.03152.

[57] D.-Q. Zhang, *Quotients of $K3$ surfaces modulo involutions*, Japan. J. Math. (N.S.) **24** (1998), no. 2, 335–366.